Le-Math

Learning mathematics through
new communication factors

Manual of Scripts for MATHFactor

Lifelong Learning Programme

Le-MATH

Learning mathematics through new communication factors 2012-2014

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Preface

The European Project Le-MATH developed a methodology for learning mathematics through new communication factors and one of them is called MATHFactor. The MATHFactor is an approach for explaining a mathematics idea orally in short time (usually 3 minutes) to a non-expert audience. In the practice of the classroom the teacher many times has this role or pupils are asked to perform such a role as a process of learning and comprehending mathematical ideas.

The MATHFActor method is expected to be performed by pupils of the ages 9-18 in different themes of mathematics including theory, applications, history of mathematics and more.

The partnership of the project Le-MATH has developed a set of MATHFactor scripts for different topics of mathematics and they are presented in this publication. The age group is also suggested but this depends also on the educational system and curriculum used in the country applied.

The scripts were analyzed in a systematic way for use by teachers and pupils. The analysis is listed as an annex in the publication "MATHFactor Guidelines for teachers and students" appearing in printed form and on the project site www. le-math.eu .

This manual of scripts is useful for teachers and pupils who would like to use them indeveloping a MATHFactor presentation or to prepare for the MATHFActor competitions. The manual is expected to be one of the materials used during the Le-MATH training course, developed by the project Le-MATH.

Dr Gregory Makrides

project coordinator and partners*

Contribution for the preparation of these Guidelines

The Guidelines are the outcome of the collaborative work of all the Partners for the development of the Le-Math Project, namely the following:

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Cyprus Mathematical Society (CY- Gr. Makrides, A. Philippou, C. Papayiannis, A. Charalambous, S. Christodoulou)) along with 12 partners from Cyprus, Greece, Bulgaria, Romania, Austria, Sweden, France, Spain, Czech Republic, Belgium and Hungary.

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Table of Contents

1. A beautiful trip to the beauty of Φ

I have stood numerous times in front of the mirror and wondered:

"What is it that makes me so beautiful and attractive?"

So, I decided to take a measuring tape and start to measure (student shows measuring tape).

I measured my height, the height of my shoulder, the height of my legs and so on. You will not believe what I came up with.

By dividing my height by the distance from the floor to my navel, I found 1,618.

By dividing the distance from the floor to my navel and from my navel to the top of my head, I found 1,618.

I also measured the length from my shoulder to the tip of my fingers and from my elbow to the tip of my fingers and divided the two lengths. Surprisingly, once again, I found 1,618.

This beauty (student shows herself) cannot be derived from luck.

So, let me make some more measurements on my head.

The distance from my chin to my eyes divided by the distance from my chin to my nose is again 1,618.

The distance from my chin to the top of my head divided by the distance from my chin to my eyes is that amazing number of 1,618 again.

Is it possible? Is this a coincidence?

This cannot be happening only to me!

Human analogies are a miracle. Therefore, the ancient Greek mathematicians called that number a divine number and its symbol is Φ, in memory of the great sculptor Fidias.

(The student picks up a rose)

Look at this rose, look at this beauty, is it luck that makes it so beautiful?

(The student picks up a coral)

Is this coral so magical just by luck?

Do you think that it is by accident that the Parthenon impresses people generation after generation?

All of you, when you have something to construct in the future, or when you see something beautiful, take a measuring tape, measure the lengths and make your calculations in order to come up with a conclusion.

Something goes Φ.

2. A Circle is a Circle

Most people have heard the line: **"A rose is a rose"**. This famous quotation comes from Gertrude Stein and is often interpreted as meaning "things are what they are".

However, times are changing and we need to ask: "Is a rose a rose?" The answer can be found by focusing on a mathematical problem – the ratio of a circle – as it has developed throughout the centuries.

In very early times, people tried to approach the phenomenon of a circle. Due to its paragon of perfection, the circle was considered to be of great value. The ratio of a circle's circumference to its diameter was very important regarding the calculation of the length of a metal mount of a wheel, a sheaf garland or for calculating the capacity of a barrel.

In the Bible we can find evidence relating to the calculation of a circumference (Book of Kings, Chapter 7, Verse 23). King Salomon assigned his locksmith Hiram from Tyros to produce a round water basin for the temple of Israel: "*Then he made the sea of cast metal. It was round, ten cubits from brim to brim, and five cubits high, and a line of thirty cubits measured its circumference*".

This is the first description of the ratio of a circle's circumference to its diameter, calculated to the number 3. We can suppose that there are inexact measurements or mistakes in making the calculation, although the number 3 was also used in Ancient China.

In Egypt, Ahmes' book of mathematics, written around 17BC, gives a much more exact ratio. Instead of 3 $({}^{16}/_{9})^2 \approx 3,1605$ was used.

In ancient Babylonian, the number 3 was used early on. Later, they used $3 + {}^{1}/_{8} = 3,125$.

In India, architects used the Silbasutras. This was a guidance document for "art of measurement." In this document, we can find $({}^{26}/_{15})2 \approx 3,0044$. The mathematician Aryabhata defined the ratio in the 6^{th} century as exactly 3, 1416.

The Ancient Greeks cultivated the fundamental knowledge of circles.

Around 250 BC, Archimedes tried to find an algorithm for calculating the exact value of π. He used polygons as a geometrical approach. His approach was dominating the field of mathematics for over 1000 years and, as a result, π is sometimes referred to as “Archimedes’ constant”. This approach involved drawing a regular hexagon inside and outside a circle, and successively doubling the number of sides until reaching a 96-sided regular polygon. He computed upper and lower bounds and defined that the ratio of circles has to be smaller than $3+{}^{10}/_{70}$ but bigger than $3+{}^{10}/_{71.}$ This means:

$3{,}1408450 \approx 3+{}^{10}/_{70}\ \ \pi < 3+{}^{10}/_{71} \approx 3{,}1428571$

His last and undoubtedly best definition is:

$3 + 8915/62991 < \pi < 3 + 3{,}1415281$ $(3{,}1415281 < \pi < 3{,}01416349)$

As time went on, up to the Middle Ages, there was total stagnation.

But let us have a look at China: Zu Chong-Zhi lived from 430 to 501 and found the first 7 decimal numbers of π. 3, 1415926 < π < 3, 01415927. Additionally, he knew the approximation fraction using 355/113 ≈ 3,141592920.

Dschamid Mas’ud al-Kaschi, a mathematician from Persia, computed 1424 π to 16 decimal numbers by using 3.2^{28} –gon.

In the 16th century Europe, Ludolph van Ceulen computed the first 35 decimal numbers of π by using Archimedes’ method but using a 2^{62}-gon. This result is named “Ludolph’s Number”.

The end of this progress started with Francois Viète in 1593. The calculation of π was revolutionized by the development of an infinite series of techniques. Nowadays, we can find the calculation of π up to 10 million decimal numbers on the internet (http://www.pibel.de/), i.e. 3. 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2 6 4 3 3 8 3 2 7 9 5 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4 4 5 9 2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 9 8 6 2 8 0 3 4 8 2 5 3 4 2 1 1 7 0 6 7 9 8 2 1 4 8 0 8 6 5 1 3 2 8 2 3 0 6 6 4 7 0 9 3 8 4
4 6 0 9 5 5 0 5 8 2 2 3 1 7 2 5 3 5 9 4 0 8 1 2 8 4 8 1 1 1 7 4 5 0 2 8 4 1 0 2 7 0 1 9 3 8 5 2 1 1 0 5 5 5 9 6 4 4 6 2 2 9 4 8 9 5 4 9 3 0 3 8 1 9 6 4 4 2 8 8 1 0 9 7 5 6 6 5 9 3 3 4 4 6 1 2 8 4 7 5 6 4 8 2 3 3 7 8 6 7 8 3 1 6 5 2 7 1….

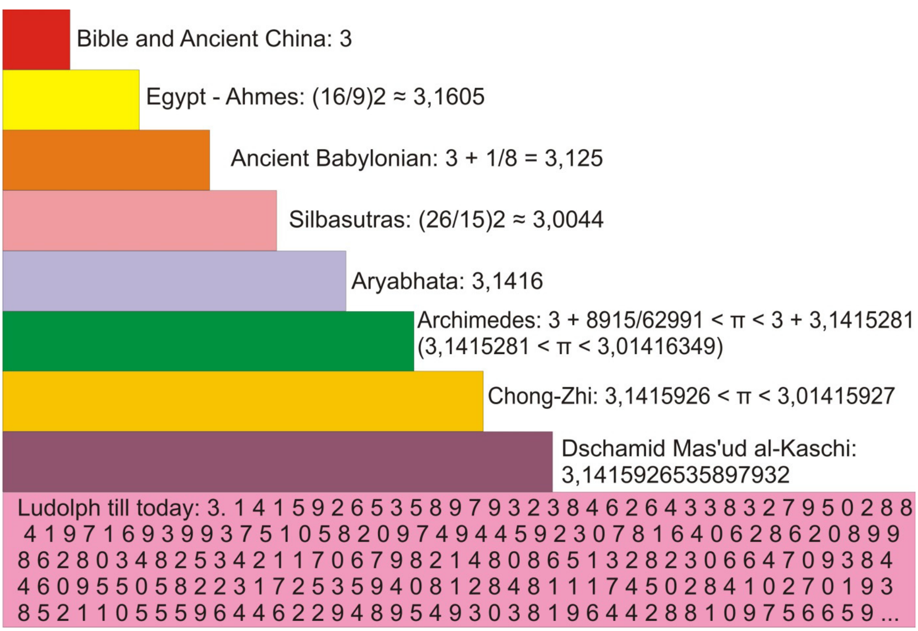
Now let us prove if a rose is a rose:

We'll use the example of King Salomon, i.e. a sea of cast metal. It was round, ten cubits from brim to brim.

The Bible, Ancient China, Ancient Egypt: 10m x 3 = 30m
Egypt: Ahmes Book of Mathematics: 10m x 3,1605 = 31,605m
Babylon Sylbasutras: 10m x 3,0044 = 30,044m
Aryabhata: 10 m x 3,1416 = 31,416m
Archimedes: 10m x 3,14159 = 31,4159m
Chong-Zhilved: 10 m x 3,141592920= 31,41592920m
Dschamid Mas'ud al-Kaschi: 10m x 3,1415926535897932 = 31, 415926535897932 m
Ludolph until present: for this example we do not get any measurable influence to the result.

My conclusion is that the more we explore and hunt for exact results in mathematics the more we can see that "a rose is a rose".

Table to be used during presentations:



3. A trip to the moon

One summer night I was sitting outdoors and was looking at the stars and the moon. It's amazing to see all these stars. You can see infinity right in front of you. The only thing you have to do is try to count the number of stars. Well, that's infinite.

How far are all these stars from the Earth?

How far is the moon from the Earth?

With a small search on the internet, I found that the moon is about 384 400 km away. Wow, that's pretty far away.

Just to give you an idea in order to understand how far that is, let's consider travelling by car from here to the moon. If the speed limit is 100km/h then I will need about one year of non-stop travelling for a round trip.

If I had an imaginary ladder that had an infinite number of steps and each step had a 20 cm distance from the other, then I would need to climb about 2 billion steps in order to go from the Earth to the moon.

Well, I think it is too much so I need to find another way for my trip.

Now, if I had a piece of paper that was 0.05 mm thick lying on the floor, (just like an A4 piece of paper), and I folded it, it would become 0.1mm thick. If I fold it 6 times then it will become more than 3 mm high. Every time I fold the paper, the height will be doubled. If I keep folding it for 43 times, then the paper will be $2^{43} * 0{,}052^{43} * 0{,}05$ which is more than 400 000 km high. That will be more than enough to reach the moon.

Wow! So, the only thing that I have to do is fold an A4 piece of paper for 43 times and there I am! On the moon.

Ok. Let us get back to reality.

The paper folding is only imaginary, or, as the mathematicians say, it is theoretical.

In reality, If you try to fold an A4 piece of paper then you would be surprised to see that you can fold the paper only 6 times or 7 times if you are very strong.

This means that we will be able to travel about ...1/2 cm.

As a result, according to the mathematics theory, the only thing you have to do to make that trip is to find a piece of paper that is large enough and then fold it for 43 times.

4. Busy as a bee – mathematics and mysteries of nature

Student walks on stage carrying three pictures; one of a honeycomb, one of a turtle's carapace, and one of Giant's Causeway.

Have you ever seen a honeycomb? When you look at it you can't help wondering why every cell is a perfect hexagon. *(Showing the picture of the honeycomb)*

By this I mean that bees could build honeycombs from rectangles or squares or triangles, but for some reason they choose hexagons for building their honeycombs.

There are several other examples of hexagons in nature, such as the scutes of a turtle's carapace *(showing the picture of the turtle)* or the basalt columns of Giant's causeway in Ireland (*showing the picture of Giant's causeway)*. There must be a reason for this other than the fact that they are cool to look at...

But why do bees prefer the shape of a hexagon? And not any hexagon, no – they like perfect hexagons with equal length on all six sides.

This is a question which has intrigued people for a very long time. In 36 B.C., a Roman scholar named Marcus Terentius Varro came up with an answer called "The Honeybee Conjecture." Varro thought that there might be a reasonwhy bees behave this way. Could it be that a honeycomb put together by hexagons can hold more honey? Or maybe a hexagon honeycomb requires less wax for building?

The main function of a honeycomb is to store honey. It takes many, many hours and flights for the bees to collect nectar from flowers in order to make honey. It is easy to assume that they want the best available storage for their product.

So, how do you build the safest, strongest possible honeycomb and as fast as possible?

If the honeycomb was made up of any random shape, this would mean that the bees had to work one at the time, making one cell and then making the other one in order to make a tight structure without gaps. Each cell would have to be customized to fit to the other.

Nevertheless, anyone who has watched bees making a honeycomb (if you haven't, check it out on youtube!) will have known that this is not the way bees work. No, they are all busy little bees working at the same time building perfect hexagons. In this way nobody is left idle and all the cells fit perfectly together – a lot like a jigsaw puzzle.

So, our initial question was, what is the reason for which bees choose to build honeycombs made by hexagons? Why not any other shape? Well, some shapes are not good, such as spheres and pentagons, since they would create gaps needing extra wax/building material for patching between them. It is essential to build a compact structure, since you want to save wax. Wax takes a very long time for bees to produce and they do not want to waste more energy than needed. They are busy and economic little workers!

Some of you are probably thinking now that bees could choose to build triangles or squares instead of hexagons. All of these three shapes can fit together on a flat surface without leaving gaps.

So, once more, why do bees prefer hexagons?

Back to Marcus Terentius Varro. Marcus proposed that a honeycomb made out of hexagons is probably a tiny bit more compact than a structure made up by squares or triangles. He couldn't prove it mathematically; it was just a conjecture, a guess; but this is what he thought.

More than two thousand years after Marcus Terentius Varro had proposed his conjecture, a mathematician at the University of Michigan, Thomas Hales, proved him right. In 1999, Hales produced a mathematical proof showing that a hexagonal structure is more compact than any other structure.

Now, one can only wonder how bees know this…

5. Camping

(Student brings with him two sticks of the same length about 80cm each)

John and Mary went for camping.

After they collected all the necessary equipment the drove off to a camping place.

John placed his tent here *(holding the stick horizontally and showing the one edge of a stick*) and Mary placed her tent here *(showing the other end of the stick).*

They then needed to find the right spot in order to place the only water tank they had.

In order to be fair to each other, they decided to place the water at a point that was located at the same distance from the two tents.

John said: “We could place it right in the middle *(shows the middle of the stick)* of the two tent.s. In this way, it will be fair for both of us. However, if we want to move from one tent to the other during the night, we will probably stumble on the tank.

Mary replied: “You are right. Is there another point where we could place the water so it’s not in our way and still be at the same distance from the two tents?”

John said: “Well, let me see”. *(after a few seconds)* “I think that if we put it here *(shows a point right above the midpoint)* it will be equidistant as well”.

Mary added: *(shows a point below the midpoint)* “It will also be fine here…”

John replied: (shows more points above and below the midpoint) “…and here, and here, and here and here and in every point of this line *(moving his hand up and down showing the perpendicular bisector)*

Mary: “Hey, you know what? You are right. That line is called the perpendicular bisector *(places the other stick to represent the perpendicular bisector)* and, as we learned in geometry, every point on the perpendicular bisector is equidistant from the edge of the segment that it bisects.

John: “Ok, let’s place the water now and forget about geometry”.

The next day their friend Mike came for camping as well and placed his tent at a clean spot. However, the water was too far away from him so they wanted to find the right place for the water in order to be at the same distance from the three tents.

John said to Mike: “Well, as we said yesterday, if we placed the water anywhere on this line *(shows the second stick)*, then Mary and I would be ok. So, we only have to find a point on this line that is the same distance from you as well”.

Mary: “Well, if we find the perpendicular bisector of Mike’s tent and mine, then all the points on that line will be equidistant from our tents. Therefore, the only thing that is left is to find the point where the two perpendicular bisectors meet and we are all set”.

Mike: “You are right. That point is called the circumcenter and is the center of a circle that passes through the vertices of a triangle”.

Mary: “It’s amazing how geometry helps us even for camping”.

6. Creation of conics

The student talks about how he/she has drawn a line and a point on a sheet of paper and how he/she has tried to fold the paper in such a way that the point is placed on the line in the folding process. While talking, the student keeps repeating the whole process of folding.

After some time he/she notices that the lines create an interesting shape. He shows fig. 1.

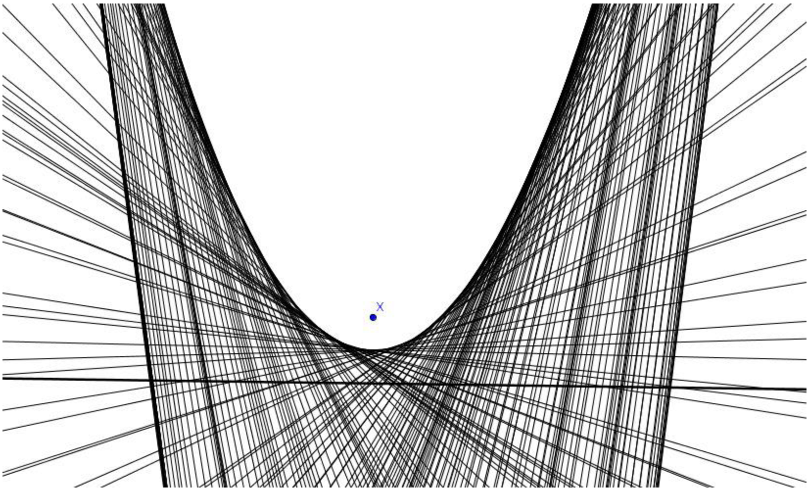


Fig1. Folding the paper - result

The points where the folds intersect eventually form a curve. This curve is a parabola (in case of an upper secondary student, it is possible to add that the initially drawn point is the focus of the parabola and the initially drawn line is its directrix).

Then the student goes on to speak about other conics – the hyperbola and ellipse; he/she tries to recreate them by folding a sheet of paper. Instead of the initial straight line, he/she draws a circle on the paper (the sheet of paper with a circle and a point may be prepared beforehand and only taken and shown to the audience) and again repeats the folding procedure several times as above. For the result, see fig. 2.

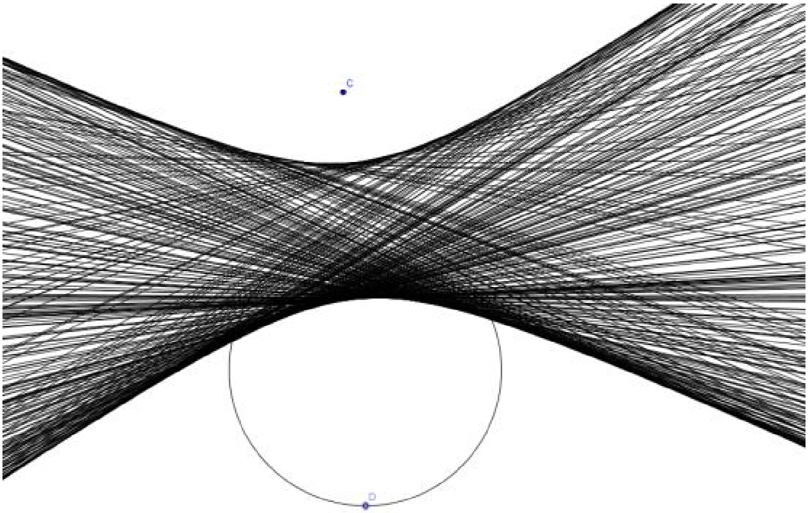


Fig.2 Hyperbola

Finally, he/she moves the initial point inside the drawn circle and gets an ellipse (see fig. 3).

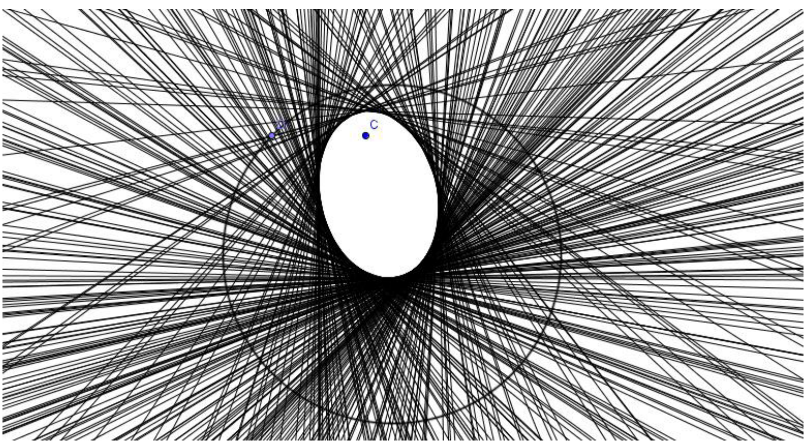


Fig3. Ellipse

The student says that he /she was so enthusiastic with the resulting pictures that he/she tried to learn more about their history and properties. The student firstly consulted Wikipedia where a lot of information was found and cited the information that he had found very interesting.

For example: It is believed that the first definition of a conic section is due to Menaechmus (died 320 BC). Among those who worked with conics, we can find such big names like Euclid (fl. 300 BC), Archimedes (died c. 212 BC), Apollonius of Perga (died c. 190 BC), Pappus of Alexandria (died c. 350 CE).

An instrument for drawing conic sections was first described in 1000 CE by the Islamic mathematician Al-Kuhi. After translating Apollonius' work into Arabic, Persians found important applications and extensions of the theory (e.g. Omar Khayyám used conic sections to solve algebraic equations).

In Europe, we can also find important names that extended the theory further, such as Johannes Kepler (1571-1630), Girard Desargues (1591-1661), Blaise Pascal (1623-1662) and René Descartes (1596-1650).

The student provides a link to the webpage address where more information can be found (see below):

http://en.wikipedia.org/wiki/Conic_section

7. Covering a chess board with dominoes

The student brings 6 figures of chessboards as shown below.

Today, we will see the different ways in which a sociologist and a mathematician accept a statement as being true or false.

On the one hand, we have the sociologists who very often conduct research. Every now and then you hear on the radio or read in a magazine about a researcher that came up with a certain conclusion. Many times these conclusions are contradictory to other studies. For example, one day you hear about a survey which concluded that women find bold men attractive, while the next day you hear about another survey which concluded that women like men with long hair.

On the other hand, you have the mathematicians. When they observe that something is true for a number of cases, they then try to come up with a proof that no one will be able to reject.

I currently have a hobby which is covering different surfaces with my dominoes without any dominoes overlapping. The other day I found a chessboard like this one (student shows the chess board to the audience) and realized that each domino covers exactly two squares on the chessboard. I was therefore making some hypotheses.

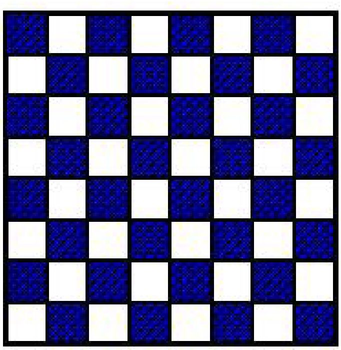


Fig.1

Case 1

Is it possible to cover all 64 squares with my dominoes?

Now, if I were a sociologist, I would get the dominoes and start placing them on the chessboard and trying to cover it. If I succeeded, I would come up with the conclusion that it is possible to do so.

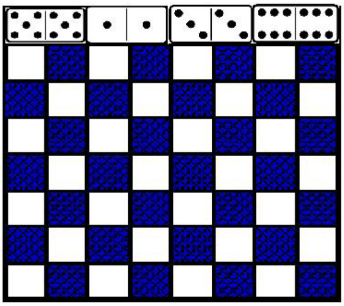


Fig.2

However, if I was a mathematician, I would say that each row has 8 squares, therefore 4 dominoes would be needed to cover a row. By repeating this for every row, the chessboard can be fully covered.

Therefore, it is not only possible but it is certain.

Case 2

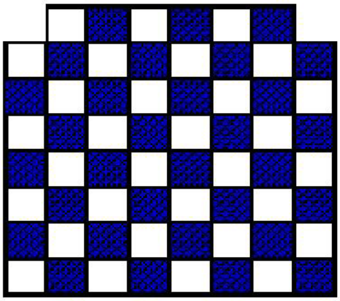


Fig.3

If the 2 corners from the top row of a broken chessboard are missing, could I cover all the squares with my dominoes?

As a sociologist, I would get a number of dominoes and try to cover the chessboard. If I succeeded then I would conclude that it is possible.

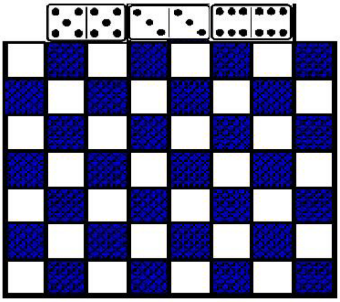


Fig.4

As a mathematician, I would say that if I place the dominoes horizontally, then the top row which has 6 squares to be covered will need 3 dominoes and the rest of the rows will need 4 dominoes each. Just as in the previous case, it is therefore certain that it can be covered.

Case 3

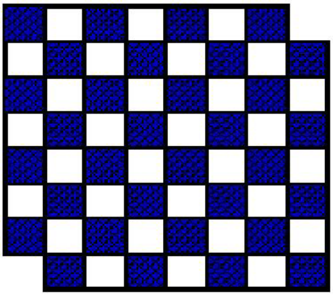


Fig.5

If the 2 diagonal corners are missing from the broken chessboard, is it possible to cover all squares with my dominoes?

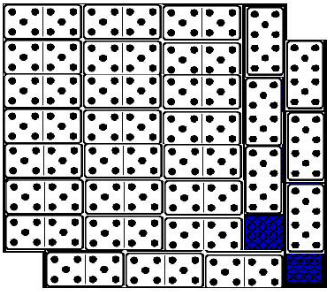


Fig.6

As a sociologist again, I would take my dominoes and try to cover the squares. I would try and try many times and after a long time I would come up with the conclusion. Even though it seems possible since I am left with two empty squares, we don't yet have such evidence to support this opinion.

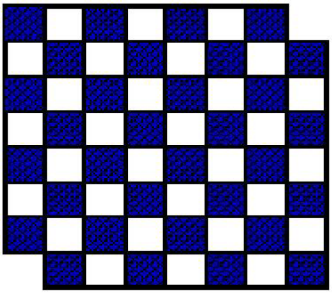


Fig.5

As a mathematician, I would say that the two missing squares are both white. So I have 30 white squares and 32 black squares left. Each domino covers one white and one black square.

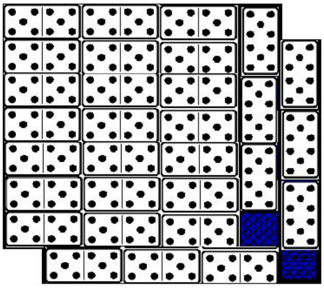


Fig.6

If I place 30 dominoes on the board, I would cover 30 white and 30 black squares leaving two black squares empty. Since there are no two adjacent squares with the same colour, it is certain that it cannot be covered.

Therefore, for reaching a quick conclusion with logical proof, it is always better to use the mathematical approach.

THANK YOU

8. Curry's Triangle

Curry's Paradox was invented in New York City in 1953 by Paul Curry.

The student could show two paper documents where the two following triangles are drawn.

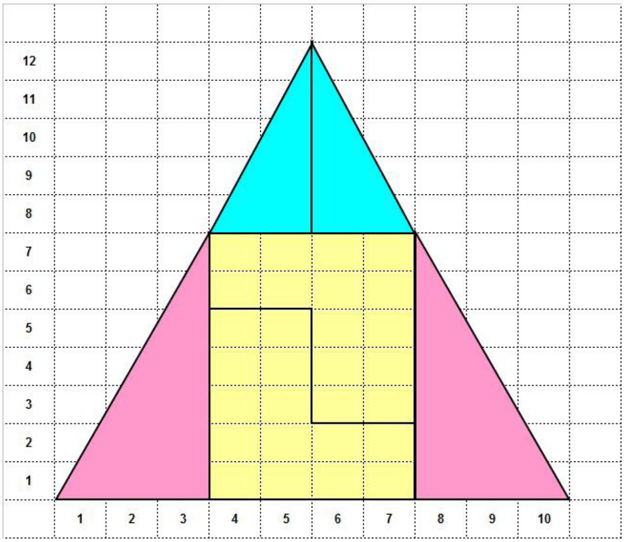


Fig.1

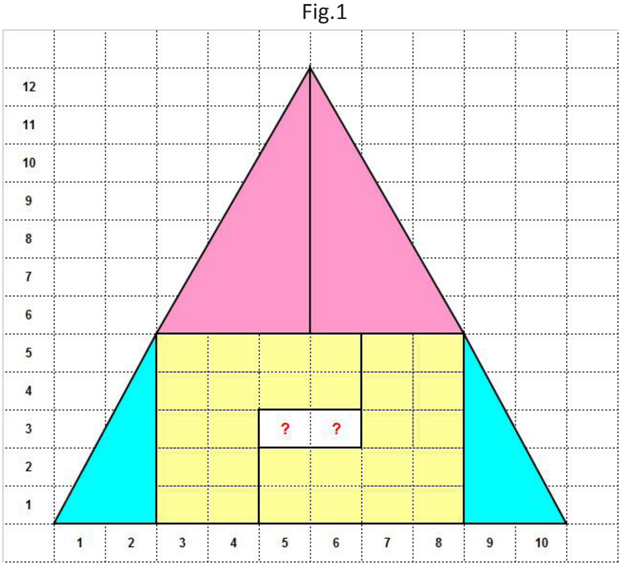


Fig.2

He can also present a little double magnetic board with wooden magnetized elements to be built: 2 blue triangles, 2 red ones and 2 yellow" L" shapes. The same grid is drawn for both parts.

On the first board, he will arrange the 6 elements in order to get the first triangle. A triangular frame can then be drawn on this first board.

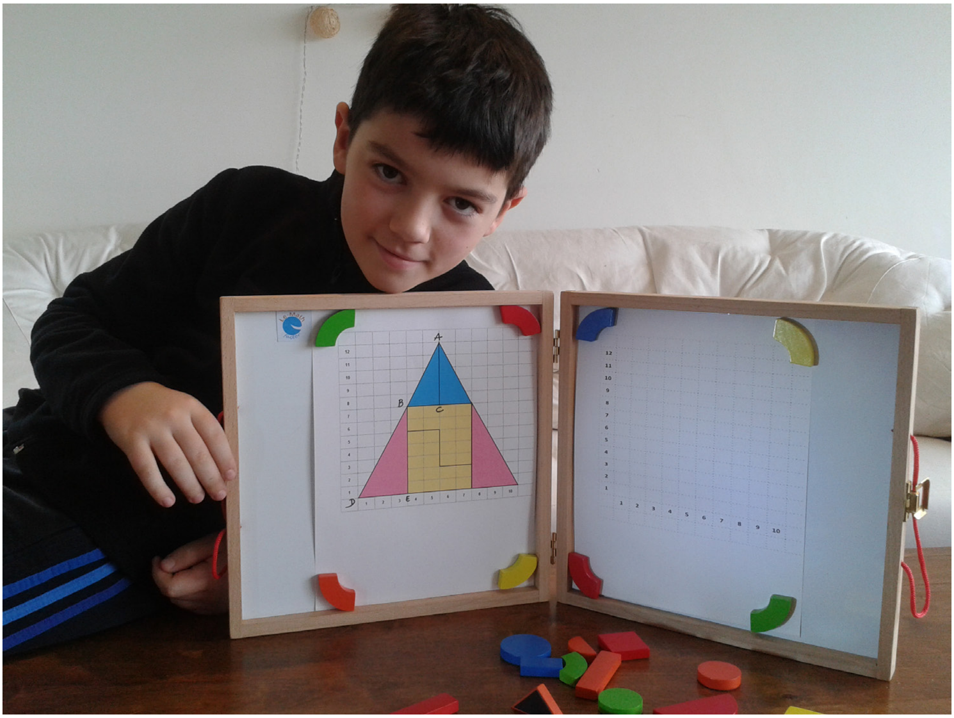


Fig.3

As you can see, we have built an isosceles triangle, with a base of 10, a height of 12 and with all 6 elements completely filling the triangle.

Let us now take all these elements, re-arrange them in another way and build a triangle on the other board.

We can see that this new triangle is an isosceles triangle too, with a base of 10 and a height of 12 as well.

You can see that these two triangles have the same area, although two squares are missing in the second one!

Is it a magic trick?

It appears to be, but this can be explained through mathematics.

Let us compare these types of angles! There are two parallel lines, (BC) and (DE), crossed by a transversal line, so that these two angles are corresponding angles!

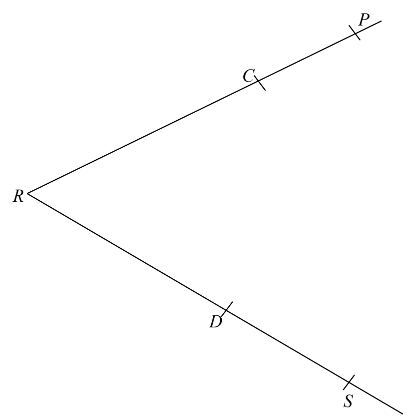
I also know that when two parallel lines are crossed by a transversal line, they determinate equal corresponding angles! However, in our case, our computations led us to the conclusion that the two angles were unequal!

What is the problem? In fact, the two lines are parallel lines, because they are the lines of the grid..., so the only solution is that the transversal line is not a straight line! The three points A, B and D are not aligned as they seem to be. The transversal passing through A, B and D does not exist!!

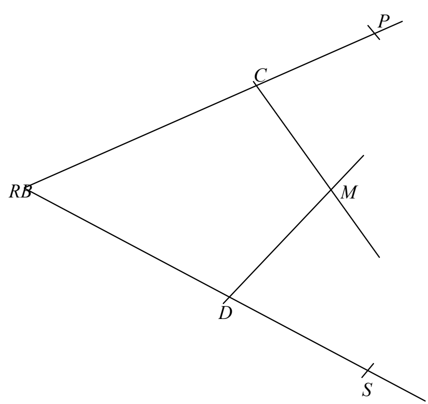
In fact, there is no magic, only mathematics!

9. Find the mistake

Yesterday, I found a competition on the Internet. There was a problem that I will now share with you. Follow me step by step: Draw an arbitrary angle *PRS* and choose the point *C* on the side *RP* and the point *D* on the side *RS*.



Draw the line perpendicular to *RP* in the point *C* and the line perpendicular to *RS* in the point *D*. These two lines intersect at the point *M*.



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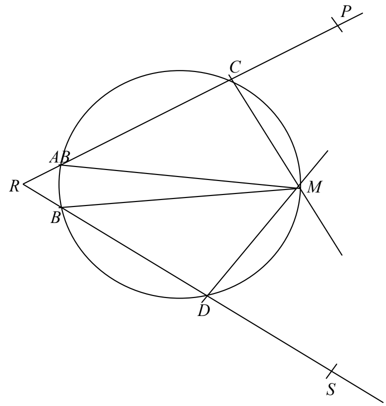
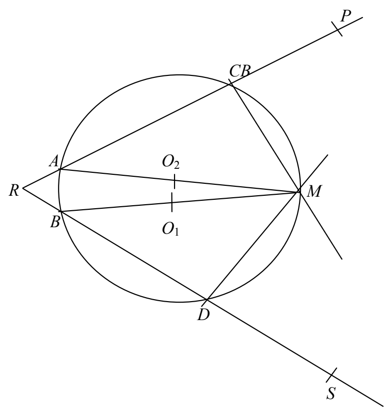
The points *C, M, D* form a triangle; draw the circumscribed circle to this triangle. The circle intersects the sides in two points; let us label them *A, B*. Connect the points *A, B* with the point *M*.



The angle *BCM* is a right angle at the circumference; therefore, the chord *BM* is a diameter of the circle. The same is valid for the angle *ADM: AM* is a diameter of the circle. It follows from here that the centers O_1, O_2 are both centers of the circle – the circle has two different centers.



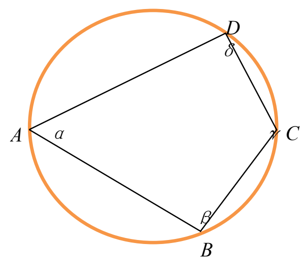
The author's question was: Have you ever seen a circle with two different centers like the one that we have just constructed? Does such a circle exist? The author promised to award a prize to the first person who would send him the explanation of the situation.

I wanted to win and I knew that a circle with two different centers couldn't exist. There must be a mistake! But where is it? I tried to find it. I went through the construction several times and could not find it.

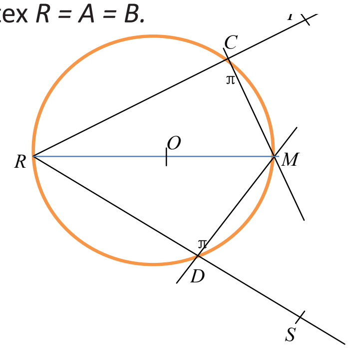
Finally, I checked our textbook and tried to find information about quadrilaterals and I succeeded in solving the problem:

What we have here is a so-called cyclic quadrilateral. Do you know what it is?

A Cyclic quadrilateral is a quadrilateral whose vertices all lie on a single circle and these quadrilaterals have the following property: A quadrilateral is cyclic if and only it holds $\alpha + \gamma = \beta + \delta$ $(= \pi)$ where α, γ and β, δ are opposite interior angles in the quadrilateral = see the picture below.



In our case $\beta = \delta = \pi$, therefore the condition is fulfilled. The circle intersects the sides of the angle in its vertex $R = A = B$.



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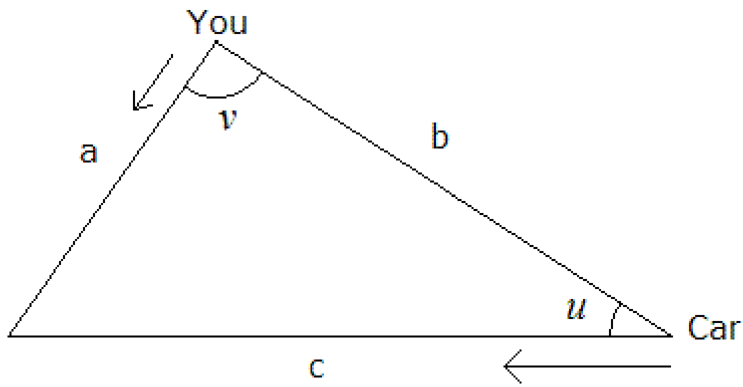
I sent my explanation of the error to the author. Now I have to wait: Did I win or not? Was my explanation correct? I guess yes. But was I the first to find it? Cross your fingers for me.

10. If you want to cross the street

When I was a kid, I learned that when crossing a street I should walk perpendicular to the street, so as to minimize the time I spent on the dangerous street. It made sense to me that one should try to minimize the danger, but I did not feel that the perpendicular path was the optimal way to solve the problem, especially if there were cars coming at a distance only from one side. Since then, I have often thought, that when crossing a street, one should walk a little bit in the direction away from the approaching cars. Now I have shown it mathematically, and I know which direction one should walk in.



You should rather walk perpendicular to the line connecting you and the car, and not perpendicular to the street.

The task is to minimize the **ratio a/c** since **a** and **c** are the paths that you and the car respectively traverse before you meet. Moreover, if that ratio is minimized we have found the path that allows the lowest possible speed for you.

We therefore want to find the minimum of **a/c** when we vary the angle **v**. Sides **a** and **c** vary with **v**. Side **b** is constant, and angle **u** is constant.

The Law of sines tells us that **sin(v)/c = sin(u)/a**. So **a/c = sin(u)/sin(v). Sin(u)** is constant, so **a/c** is minimized when **sin(v)** is maximized, which happens when **v=90** degrees, i.e. you walk perpendicular to the line **b** connecting you and the car.

11. Logarithm, i.e. arthmetic locus...

The important geographic discoveries of the 15th and 16th centuries were possible due to transatlantic navigation. Unlike the Mediterranean navigation where the sailor does not lose sight of the shore for too long, transatlantic navigation implies months of isolation with no other guiding marks but the stars.

The fact that their position in the sky at a certain moment could precisely determine the position of the ship had been known since ancient times. However, this connection was made through calculus, which implied multiplication of large numbers, which could last for days.

Undoubtedly, since the lives of the people on board, the ship-owners' fortunes as well as the naval power of the West-European states depended on the results of the calculus, a number of influential people decided to put pressure on those who could achieve fast mathematical modes.

This was the historical context in which logarithms, regarded as a wonder of mathematics, came to life. What was really exciting was their amazing capacity of turning multiplication into addition. It was too good to be true.

Briefly, the mode starts from the well-known property of the multiplication of degrees with the same base, which states:

$$a^m \cdot a^n = a^{m+n}$$

Neper had a valuable idea when he inferred that this formula, which turned a multiplication into an addition, was the instrument he needed.

Thus, if we have to multiply two big numbers X and Y, a solution would be to have a table with many numbers written as degrees of a number a conveniently chosen.

Then it is quite easy. The degrees m and n can be found in the table, for which:

$x = a^m$ **şi** $y = a^n$,

Add the degrees (m+n) and search in the table for the value of:

$\mathbf{z = a^{m+n}}$

Certainly, z is the value of the product x and y.

Let us illustrate an example:

Suppose we have to multiply the numbers 1024 and 4096.
We choose **a = 2** and suppose that we have the table in which each number up to a very big number N is written as a degree of 2.

After a quick rundown of the table, we notice that $\mathbf{1024 = 2^{10}}$ **and** $\mathbf{4096 = 2^{12}}$
We add 10 +12 = 22, then we find in the table that $\mathbf{2^{22} = 4194304}$

It means that **1024 · 4096 = 4194304**

And this is it.

These degrees m and n, to which a is to be raised in order to get x and y, have been named the logarithms of x and y in base a (the translation for logarithm would be "arithmetic locus" because in our examples, 10 and 12 are the "arithmetic loci" of the numbers 1024 and 2096 in the table that represents them as degrees of 2).

Is that all?

Certainly not. If it had been so, the solution would have been given a lot earlier. The big problem is found in the calculus of the table in use, which has proved to be too difficult for the numbers that are not full degrees of 2 (such as 3 or 170111 or any other number, except some lucky ones such as 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, ... and so on, each number being obtained from the previous one multiplied with 2).

Nesper's brilliant idea was to express the numbers as degrees of a number slightly different from 1, instead of degrees of 2. He therefore chose 0.9999999.

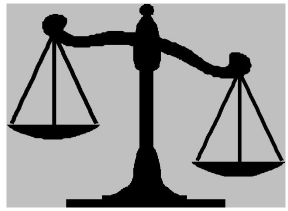
As the tables were difficult to use, a classical tool was designed: “the slide rule”. This tool had been essential for every engineer for 300 years until the development of the numerical calculators in the 1950s. Owing to today’s calculators, people have long forgotten the charming logarithms.

However, it is not this use of the logarithms that the mathematicians are fascinated with.

Lord Neper’s discovery hides a luring mathematical mystery…

12. The ideal number of weights





1g

3g

9g

sugar

The student uses a two-sided balance, weights of 1 g, 3 g and 9 g, one sugar-bowl with sugar.

Our task is to weigh a mass of 1 g by the balance. Obviously, we need a weight of 1 g. A mass of 2 g could be weighed in several different ways: using two weights of 1 g each, using only one weight of 2 g but there exists a third way, which is quite curious – using two weights of 1 g и 3 g respectively. Note that 3-1=2. Thus, in the third case we put the weight of 1 g on the plate with sugar. With other words, if the weighing is realized by a two-sided balance, then we can put the weight of 3 g on one of the balance plates, while on the second plate we can put sugar and the weight of 1 g. (*The student demonstrates the weighing using sugar from the sugar-bowl and the two weights of 1 g and 3 g.*) In case of equilibrium the sugar under weighing is exactly 2 g. There is a fourth way of weighing: first weigh 1 g, and after that weigh 1 g for a second time. Note that in the tasks we consider such a weighing is not allowed, i.e. the necessary quantity of sugar should be weighed once (unrepeated weighing). We want to find the minimal number of weights to weigh possibly greater quantities. Thus, in the example under consideration it is natural to choose the third possibility, i.e. to use two weights – one of 1 g, and a second weight of 3 g. And now by means of the two weights we can weigh all quantities from 1 g to 4 g.

Let us go further. In order to weigh 5 g we need an additional weight. Since 5+3+1=9, the optimal situation is to take a weight of 9 g. Now 5=9-3-1 , 6=9-3, 7=9-3+1, 8=9-1, 9=9, 10=9+1, 11=9+3-1, 12=9+3 и 13=9+3+1, i.e. with weights of 1 g, 3 g and 9 g we can weigh all quantities from 1 g to 13 g. (T*he students demonstrates only few of the cases using the sugar-bowl.*)

The next quantity is 14 g and it is obvious that we cannot realize it by the available weights only. But 14+9+3+1=27 and for this reason the optimal solution is to take a fourth weight of 27g. Now, 14=27-9-3-1, 15=27-9-3, 16=27-9-3+1, 17=27-9-1, 18=27-9, 19=27-9+1, 20=27-9-1+3, 21=27-9+3, 22=27-9+3+1, 23=27-3-1, 24=27-3, 25=27-3+1, 26=27-1, 27=27, 28=27+1, 29=27+3-1, 30=27+3, 31=27+3+1, 32=27+9-3-1, 33=27+9-3, 34=27+9+1-3, 35=27+9-1, 36=27+9, 37=27+9+1, 38=27+9+3-1, 39=27+9+3 и 40=27+9+3+1. (*Here the students does not weigh. He or she just pronounces slowly the above equations or only some of them)*.

The conclusion is that using weights of 1 g, 3 g, 9 g and 27 g one can weigh all quantities from 1 to 40 g. Observations show that the successive weights are the successive powers of the number 3: $1=3^0$, $3=3^1$, $9=3^2$,. We can formulate a hypothesis: "THE MINIMAL NUMBER OF WEIGHTS WHICH ARE NECESSARY TO WEIGH ALL QUANTITIES IN NATURAL NUMBERS FROM 1 TO $\frac{3^k-1}{2}$, IS EQUAL TO *k*." All the weights in grams are $1,3,9,...3^{k-1}$ .The proof of the hypothesis is based on the fact that each natural number could be represented in a unique way in a 3-base numeral system, namely:

$$K = a_0 3^k + a_1 3^{k-1} + ... + a_{k-1} 3 + a_k ,$$

where the coefficients a_i, i=1,2,3...k are equal to 0, 1 or 2. In our everyday life we use the 10-base numeral system. The powers of 10 participate in it and the corresponding coefficients are the digits 0, 1, 2, … or 9. As we see there exist other numeral systems too: 2-base, 3-base, 4-base numeral system and so in the 3-base numeral system the coefficient α_k is the remainder of modulo 3. The result should be divided by 3 again and the coefficient α_{k-1} is the new remainder.

And so on. Note that $2.3^i = (3-1).3^i = 3^{i+1} - 3^i$, which means that in the 3-base representation of a weight from 1 to $\frac{3^k-1}{2}$ the weights $1,3,9,...3^{k-1}$ are taken with sign “+” or “-” .

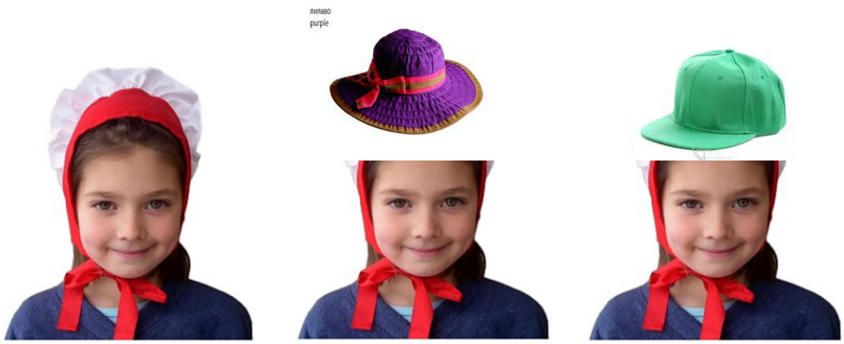
According to the formula of the geometric progression we have

$$1+3^1+3^2+...3^{k-1}=\frac{3^k-1}{2}.$$

Mathematics is extremely interesting and as we have noticed, it is extremely useful too.

13. The Little Red Riding Hood and Diophantine Equations of First Order

(The sketch is performed by the Little Red Riding Hood, who wears a red hat. She plays three roles simultaneously: besides playing herself, she performs the role of her grandmother and the role of her grandfather. The roles are changing by means of hats. As grandmother, the Little Red Riding Hood wears an old woman's hat and as grandfather she wears a cap. The performance is in front of a table with two different hourglasses and two different pails on it.)



The Little Red Riding Hood jumped out of bed and ran to the kitchen to her Grandma:

- Grandma, I want a boiled egg!

- (T*he Little Red Riding Hood puts on the old woman's hat*) O key, my darling, but to prepare it as you have a taste for eggs, I have to boil it exactly 15 minutes. Unfortunately, the clock stopped and I cannot measure 15 minutes. We have to wait for your Grandpa, who went to buy a new battery for the clock.

- (*The Little Red Riding Hood takes off the old woman's hat) But Grandma*, *why don't you use the hour-glass?*

- *(The Little Red Riding Hood puts on the old woman's hat)* I can't, because it measures 7 minutes. I cannot use your Grandpa's hour-glass either, because it measures 11 minutes.

- *(The Little Red Riding Hood takes off the old woman's hat)* O, Grandma, look how we will obtain 15 minutes. Start both hour-glasses. When the first one will measure 7 minutes, you will put the egg boiling. Since 11-7=4, after exactly 4 minutes the sand in the second hour-glass will flow out. Thus, we can measure 4 minutes. Then you will turn upside down the second hour-glass and it will measure 11 minutes more. Here you are, the problem is solved, because 11+4=15 minutes.

Delighted with the mathematical skills of her grandchild, the Grandma took in hand the proposal. Soon the egg was boiled in 15 minutes exactly and the Little Red Riding Hood ate it with satisfaction. She just licked clean, when her Grandpa entered the kitchen.

- *(The Little Red Riding Hood puts on the cap)* Let me see now what a mathematician you are! – he said and showed the two pails he carried with him. – The first pail is 8-litre, while the second one is 14-litre. I have to measure 4 litres exactly and I don't know how to proceed.

- *(The Little Red Riding Hood takes off the cap)* O, Grandpa, this task is very easy! Come on; fill in the 14-litre pail with water from the tap.

The Grandpa filled in the pail and turned his face to the Little Red Riding Hood impatiently.

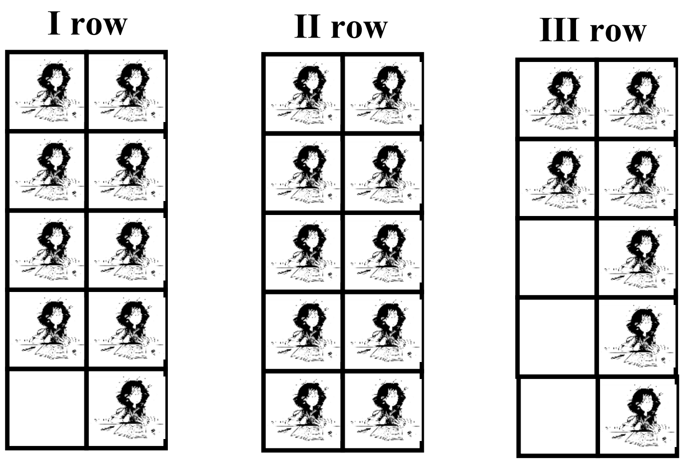
- And now, Grandpa, fill in the second pail with water from the 14-litre one. Since the second pail is 8-litre, the remaining water in the first one will be 14-8=6 litres. Further, flow out the second pail and pour the 6 litres from the first one into it. Again, fill in the 14-litre pail brimfuly with water from the tap. What is the situation: we have 14 litres in the first pail and 6 litres in the second one. Now, you can add exactly 8-6=2 litres to the second pail using water from the first one. The result is 14-2=12 litres in the first pail and 8 litres in the second one. Grandpa, it remains to pour out the second pail and to fill in brimfuly with water from the first pail. Thus, we have 12-8=4 litres in the first pail and here are your 4 litres.

- *(The Little Red Riding Hood puts on the cap)* Congratulations, my grandchild, where do you know these things from?

- *(The Little Red Riding Hood takes off the cap)* Grandpa, note, that 14.2-8.3=28-24=4. This means that we have filled in the 14-litre pail twice, while we have filled in the 8-litre pail three times. We have proceeded in such a way but in a suitable succession. Earlier I have shown to Grandma how to measure 15 minutes by means of the two hour-glasses, the first one measuring 7 minutes and the second one measuring 11 minutes. I have used that 2.11-7=22-7=15. This means that the 11-minute hour glass has been used twice, while the 7-minute one – only once. At first glance the two tasks are different but actually the principal is one and the same. The general problem is to find integers x and y, verifying αx+by=c, where *a,b* and *c* are given integers. This equality is known to be Diophantine equation, which we have studied last week in the mathematical circle. In the case of the pails the Diophantine equation is 14x+8y=4, while in the case of the hour-glasses the equation is 11x+7y=15. In both cases the Diophantine equations have solutions. In the first case it is so, because the greatest common divisor of 14 and 8 is 2, and 2 divides 4. In the second case the integers 11 and 7 are co-prime, which means that their greatest common divisor is 1. At the same time 1 divides each integer and it divides 15 in particular. The teacher in the circle told us that the Diophantine equation under consideration has solution if and only if the greatest common divisor of *a* and *b* divides *c*. It follows that the Diophantine equation 14x+8y=3 has no solution, because the greatest common divisor 2 of 14 and 8 does not divide 3. For this reason, Grandpa, if you had asked me to measure 3 litres using your two pails, I would answer that this was not possible. And this is the truth.

14. The invariant property

(The action is in a classroom with 3 rows and 5 desks in each row. The actor uses 3 plates representing the rows as shown and during the performance manipulates the plates according to the text below. Without changes the text is presented as a monologue.)



There are 3 rows in the class room. The number of the students in the first row is 9, the number in the second row is 10, while the number of the students in the third row is 7. Since 9+10+7=26, then the total number of the students in the class room is 26. What have we done? We have added the number of the students in the first row with the number of the students in the second row and with the number of the students in the third row. But we may take the students in the second row firstly, add the students in the first row after and then add the students in the third row, i. e. add the numbers 10, 9 and 7. We have 10+9+7=26. The total number of the students in the class room does not change. We may proceed in a different way. There are 11 desks in the class room with two students in each of them and also 4 desks with one student in each of them. The total number of the students in the class room is 11.2+4.1=22+4=26. We obtain 26 again. What does not change, is the total number of students, no matter how the counting is done. In mathematical language we say that the number of the students in the class room is invariant with respect to the way of counting. Invariant is one of the basic notions in mathematics.

(The actor changes the plates. Now he/she takes the following 6 plates)

1, 2, 3, 4, 5, 6	**1, 3, 3, 4, 6**	**2, 3, 4, 6**	**2, 3, 6**	**2, 3**	**1**

(The actor manipulates with the plates according to the text below).

Take the numbers from 1 to 6, i.e. the first 6 positive integers: 1, 2, 3, 4, 5 and 6. (The first plate is shown). Choose two of them, for example 2 and 5, delete them and instead of them take their difference subtracting the smaller one from the bigger one. Thus, we delete 2 and 5, replacing them by 3. The number of the positive integers decreases by 1. The remaining integers are: 1, 3, 3, 4 and 6 (the second plate is shown). Again, choose two of them, for example 1 and 3, delete them and instead of them take their difference 2. The remaining integers are four: 2, 3, 4 and 6 (the third plate is shown). Choose two integers, for example 2 and 4, delete them and replace them by their difference 2. The remaining integers are: 2, 3 and 6 (the fourth plate is shown). Further, choose two of them, for example 3 and 6, delete them and take their difference 3. The remaining integers are 2 and 3 (the fifth plate is shown). Delete them (now, the choice is unique) and replace them by the difference 1. The game is over. The remaining integer is 1 (the last plate is shown). If we start from the very beginning and change the choices of the pairs to be deleted, we can obtain another integer as a final result. The question is whether the integer 2 could be a final result. The answer is negative. Why? Along deleting, observe how the number of the odd integers is changing. At the beginning we have three odd integers: 1, 3 and 5 (the first plate is shown). There are 3 possibilities for the choice of a pair: to choose two even integers, two odd integers or two integers with different parity (one even and one odd). In the first case we should delete 2 even integers and the number of odd ones should remain 3. In the second case we should delete two odd integers and the number of the odd ones should decrease by 2. In the third case the number of the odd integers remains unchanged (deleting one even integer and one odd one we replace one odd integer by another odd integer). The observation shows that no matter what deleting is done the number of the odd integers remains the same or decreases by 2. Since the game ends by one integer, the final integer could not be even because it is not possible to eliminate all odd integers starting by 3 odd ones and keeping this number or decreasing it by 2. What happens is the following: the parity of the number of the odd integers is invariant with respect to the operation “deleting”. Thus, using the notion of invariant we have proved that the final result of the game could not be an even integer. The possible results are odd: 1, 3 or 5.

15. Egyptian Fractions

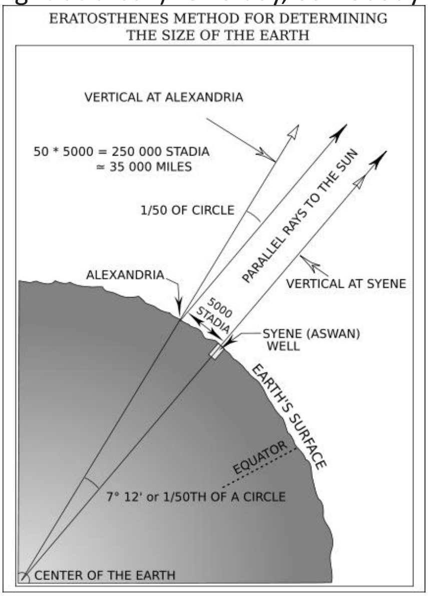
(The actor is with 3 plates on stage, as shown. The plates are used according to the text below).

$$\boxed{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}} \qquad \boxed{\frac{1}{8}} \qquad \boxed{6}$$

An ancient problem says: Share 7 loaves of bread among 8 people. "The solution is quite simple. Divide each loaf into 8 equal parts and give one eighth of each loaf to each person *(the second plate is shown)*. In such a way everybody receives seven eighths of a loaf. Of course it is possible to proceed differently: give seven eighths of the first loaf to the first person, then give the remaining one eighth of the first loaf and six eighths of the second loaf to the second person, and continue similarly. It is important that everyone will finally receive seven eighths of one loaf and this ensures the equal distribution of the 7 loaves of bread. Let us get to the heart of the matter. According to the proposed procedure everybody receives $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$ *(the first plate is shown)*, which represents $\frac{7}{8}$ of 1 loaf of bread. What is essential here is that the nominators of the fractions are equal to one, which corresponds to the division of one loaf, i.e. the dividing is loaf by loaf and not a simultaneous division of two or more loaves. This is natural. We use fractions with nominators equal to 1, which have a special name. An ordinary fraction with nominator 1 and denominator a natural number is called an *Egyptian fraction (or aliquot fraction)*. The name comes from the fact that such fractions appeared for the first time in Ancient Egypt. One of the first proofs is referred to as the **Rhind Mathematical Papyrus**, which has found its way to the British Museum in London. It probably goes back to at least 2000 BC and probably further. The papyrus was discovered in a tomb; later, in 1858, it was bought by Henry Rhind at a market in Egypt, thus bearing his name. In 1864, the papyrus was taken to the British Museum and has remained there ever since. The Rhind papyrus contains a long list of fractions with nominator 2 together with their decompositions into sums of Egyptian fractions. The way the Egyptians of 4000 years ago have worked with such fractions is evident even today. Some conjectures in Number theory are connected with them. One of the most famous belongs to the Hungarian mathematician Paul Erdös (1913–1996), which says that each fraction of type $\frac{4}{n}$ can be represented as a sum of at most 3 Egyptian fractions. It is well - known that each ordinary fraction can be represented as a sum of Egyptian fractions in infinitely many ways. The problem concerns the number of the summands to be the least possible. For example, in the case of the initial problem, the fraction $\frac{7}{8}$ is represented as a sum of seven Egyptian fractions. One way to decrease the number of the summands is to use the positive proper dividers of the denominator 8, i.e. all dividers of 8, the unity included, but without the number 8 itself. In the case under consideration, the proper dividers are 4, 2 and 1. Group the summands in the sum of the seven fractions $\frac{1}{8}$ into 3 groups: the first one containing the first 4 summands, the second one containing the next two and the third group containing the last summand. Since 4 and 2 are dividers of 8, then $\frac{7}{8}$ is represented as 4 times $\frac{1}{8}$ plus 2 times $\frac{1}{8}$ plus 1 time $\frac{1}{8}$. We obtain $\frac{1}{2}$ plus $\frac{1}{4}$ plus $\frac{1}{8}$, thus getting only 3 summands. What will happen if we reformulate the initial problem in the following way: Share 5 loaves of bread among 6 people". Now $\frac{5}{6}$ is equal to the sum of 5 Egyptian fractions of $\frac{1}{6}$. The positive proper dividers of 6 are 1, 2 and 3. Thus, we could group the first 3 summands of $\frac{1}{6}$ and the next 2 ones of $\frac{1}{6}$. We obtain that $\frac{5}{6}$ is equal to the sum of $\frac{1}{2}$ and $\frac{1}{3}$. Note that in this case the Egyptian fractions with denominator 6 disappear. It is not the same with the Egyptian fractions with denominator 8 – one Egyptian fraction with denominator 8 remains. The reason is that the sum of the proper dividers of 6 is equal to 6 exactly, i.e. 1+2+3=6 *(the third plate is shown)*. Six is a perfect number. Each number which is equal to the sum of its positive proper dividers is known to be a *perfect number*. The next perfect number is 28, because 1+2+4+7+14=28. The third perfect number is 496. There are others and all known perfect numbers are even. There is another conjecture in Number theory, namely that all perfect numbers are even, which, however, still remains unproved. It turns out that the world of the Egyptian fractions and the perfect numbers is extremely challenging.

16. How did Eratosthenes manage to calculate the circumference of the Earth 200 years BC?

More than a thousand years ago, Eratosthenes of Cyrene calculated the circumference of the Earth with a remarkable accuracy. Like the Pythagoreans, he also believed that the Earth's shape could be nothing else but the perfect shape: a sphere. Knowing this, he went one step further and calculated its circumference in a strict but elegant mathematical way. He lived in Alexandria and had understood that the sun's rays are parallel between them. (getting a flashlight and pointing it at a ball). One day, somebody told him that at Syene,



today known as Aswan, during summer solstice, the sun shone to the bottom of a well. Consequently, he assumed that the sun was at its zenith, so, in theory, its rays would pass from the centre of the Earth if he could possibly extend them (gets a plastic hoop and demonstrates with the use of sticks – see image). Thus, on this hoop that represents the Earth, this is Syene and this ray (stick), if extended, would pass from the centre of the sphere. The other rays are parallel to this. This is Alexandria and this ray passes through Alexandria. If extended, it would reach the centre of the Earth. However, this is not parallel, there is an angle formed here, an angle which one could measure using a gnomon.

And so he did and found it to be 1/50th of a circle’s circumference, or 7°12’. Now, he only had to measure the distance between Alexandria and Syene. The legend says that he sent his students to measure this distance. In any case, he found it to be 5000 stadia. Knowing that 1 stadium is 185 metres, we can find that the distance is 925 kilometres. Then he made this brilliant assumption:

If 1/50 of the circle represents 5.000 stadia, then 50/50 of it will represent 250.000 stadia or 46.250 kilometres. This was the best estimation humans had made for over 1000 years. Nowadays, we know that the circumference of the Earth is 40,075.16 km on the equator and 40,008 km at the poles.

17. Hidden Paths and Patterns

Imagine you are at a party and there are 20 people in the room, including yourself. If every two people that meet start shaking hands once with each other, is it possible to estimate the total number of handshakes that are made?

To find the solution to this problem, we need to go back, way back, inside the walls of an elementary school in the Lower Saxony of Germany, in the end of the 18th century. The classroom is full of boys who insist on not listening to their math teacher until his patience finally runs out. In order to keep them really busy for a while, he assigns to them an adding problem that should keep them occupied for the rest of the day. He tells them to add all the whole numbers from one to one hundred. That is *(we take out a paper board to demonstrate):*

$$1+2+3+4+5+\ldots+95+96+97+98+99+100$$

This is a brain-teasing problem not only for eight-year old kids of that day, but even for today's adults. But it is not a problem at all, if your first name is Carl Friedrich and your surname is Gauss. Before his fellow classmates even had the chance of writing down all the numbers from 1 to 100, he had raised his hand knowing the answer.
How did he do this?

He observed that in the series of numbers from 1 to 100, the sum of pairs of numbers from each end is 101. So he wrote down two lines: one having the numbers in their correct order, from 1 to 100 and one having them in reversed order, from 100 to 1.

(Paper board again)

$$\begin{array}{l} 1+2+3+4+5+\ldots+96+97+98+99+100 \\ 100+99+98+97+96+\ldots+5+4+3+2+1 \\ \hline 101+101+101+101+101+\ldots+101+101+101+101+101 \end{array}$$

So in the end he had 101, 100 times, equalling 10100. But in this way, each number has been added twice, creating the need to divide by 2. So the final answer is 5050.

Little Gauss, at the age of eight, saw a hidden pattern and used the pair-up and the reverse-doubling method, two basic techniques for modeling. This is not the only problem for which this methodology can be applied to. This little trick will work for any series of numbers, provided that they are evenly spaced. For example, if we have the numbers

2, 4, 6, 8, 10, 12, 14, 16, 18, 20

in order to find the sum, we only need to add the first and the last number (22) and then multiply them by 10, as we have a series of ten numbers and divide them by 2.

The same applies for the handshake problem. If everyone shakes hands once with each other, then we have each person shaking hands 19 times. Thus we have 19 * 20. However, by using this way, we counted handshakes between each pair twice, so we need to divide by 2. This gives us 190 handshakes.

The same applies for more complicated problems, like the total number of connections in a network of x computers, the total number of games among x sport teams, the total number of edges and diagonals of an n-polygon, the different pairs (2-combination) among x people or objects, and so on. The only thing we need to do to move on quickly and easily with such problems is to find the secret paths and patterns!

18. How does Santa make it?

The same thing happens every single year. We are waiting and waiting, all night long, staring out of the window. Until at some point, we close our eyes and the next thing we know is that we wake up in our bed. And under the Christmas tree, the presents are already there but Santa is already gone…

So what is truly happening? Does Santa Clause really manage to bring our gifts? And how does he make it, since millions of children are waiting for him and he only has one night?

Well, not just one night. To begin with, he has more than one night, or we might say, if you'd prefer, that he has one very long night. Taking advantage of the time difference among the various places on the planet, he starts his journey from the East where Christmas night first arrives and then he moves towards the West. This way, he has a little more than 32 hours to deliver his gifts. But how many gifts will he deliver and to how many kids?

Almost 2 billion kids live on Earth today. If in every house a family has 2.5 kids on average (!), then 800 million houses are waiting for Santa Clause every year. If the average distance between two houses is 400 metres, then Santa has to cover a distance of 320 million kilometres in 32 hours! His mission seemed impossible, until Uncle Albert and his theory of relativity jumped in to make things …easier.

According to the theory of relativity, the maximum speed that exists in the Universe, is the speed of light at 300 000 kilometres per second. If Santa's super high-tech sleigh could reach this speed, he would need no more than 1066 seconds to cover this distance. However, he would have to face some technical difficulties, as this speed is unreachable for particles containing matter of which we are all made of. This happens because nature puts a physical barrier to all things carrying mass. The more they approach the speed of light, the more their mass grows. They will never make it to 300 000 km per second, as their mass will grow and grow and will finally approach infinite mass. The exotic phenomena of the theory of relativity appear and surprise us and will never let Santa deliver his gifts, unless, a smaller speed was just enough for him to make it. If he could just reach 40% of the speed of light, his mass wouldn't grow too much and he would be able to travel the world. By having a speed of 125 000 km per second, he would need 2560 seconds to do 320 million kilometres.

And if we wanted to go the other way around and wonder what exact speed he would need in order to make it just on time, it's a matter of a simple division to get the answer. By dividing 320 million kilometres with 32 hours, we get 10 million kilometres per hour. With yet another division with 3600 (the number of seconds an hour contains), we get 2 777 kilometres per second. Spaceships go on 20 000 kilometres per second, so if NASA can do it, for Santa Clause it should be a piece of cake!

19. Lucky bet

Chevalier de Mere was a noble man and bon viveur of the 17th century. It also happens that he was a gambler. At the time, gamblers loved dice and one of the most common gambling games was to try to predict if a specific number would be included in a dice outcome. For instance, de Mere's favourite game was to bet that number six would appear at least one time in four consecutive throws of the dice.

He used to make good money out of this activity, until one day he decided to take it a step further. He started betting that a double six would appear at least once in twenty-four consecutive throws of two dices.

However, this was not usually the case. He started losing money and his reputation started changing. Now he was considered to be unlucky.

Chevalier de Mere however, was not a man to believe in luck but in math! He was fortunate enough to personally know Blaise Pascal, one of the best mathematicians of all times and he decided to ask him what was wrong with his new game. Pascal studied the problem and sent a series of letters on this to his colleague Pierre Fermat. Their written correspondence was the foundation of the modern theory of probability.

The theory of probability helps us understand a large range of things: from the odds of winning the lottery to the odds of getting struck by lightning. To find out how probable or improbable an event is, we have to think of all the possible outcomes of something and then see how many outcomes there are, in favour of what we want. A simple division of the favourable outcomes with the total number of outcomes gives us the requested probability.

So which is the case with de Mere's problem? In the first game, if the dice was thrown once, it had 6 possible outcomes. The favourable one, number six, has a probability of 1/6 to appear, while the probability of not getting a six is 5/6. If we throw the dice four times, the probability of not getting a six is 5/6 raised to the power of 4 which is almost 0,48 or 48%. So the odds were with him: he risked less than 50% in his first game, as the outcome of no six appearing had a chance of 48%.

In his second game however, he simultaneously rolled two dice with the possible outcomes being 36 combinations of two numbers. A double six has a probability of 1/36 and not getting a six has a probability of 35/36. If we throw the dice 24 times, we then have to raise this to the power of 24, which gives us 0,51 or 51%. So in his second game, the odds were against him. The outcome of not getting a double six was more than 50%.

My guess is that after knowing this, de Mere might have turned back to his first game. What's yours?

20. The sound of music

“Music is the pleasure that the human mind experiences from counting without being aware that it is counting”, Gottfried Leibniz used to say. Modern research shows that when children are trained in music at a young age, they tend to improve their math skills. So, there is undoubtedly a link here, connecting two of the most basic and most fundamental forms of science and art, i.e. math and music.

The Pythagoreans are the first known researchers of the mathematics of music, as far as musical scales and ratios are concerned. They used to believe that “all nature consists of harmony arising out of numbers”, while Plato considered harmony to be a branch of Physics now known as musical acoustics.

The legend has it that Pythagoras himself happened to pass by a blacksmith who was hitting his hammer on the anvil, producing some ringing notes that caught the great mathematician’s attention and aroused his curiosity. He decided to investigate the phenomenon and soon concluded to the very first music ratio, the ratio of the anvil. According to this ratio, a double sized anvil would produce notes with exactly half the frequency of the original notes.

Pythagoras then wondered if he could find a similar ratio in stringed instruments. He experimented and after a while he managed to prove that if he cut a string in half, the frequency of the note would double because the smaller string could oscillate two times faster. Doubling the string would result to reducing the frequency in half, just like doubling the anvil size. Finally, he discovered that by resizing the string he could produce all the notes of the musical scale in certain ratios.

Nowadays, we know that a musical scale is a series of pitches. The most important scale in Western music is the diatonic scale C-D-E-F-G-A-B (do – re – mi – fa – sol – la – si). Two consecutive Cs include an octave and the higher C will have the exact double frequency. As a result, the lower one is singing at 262 Hz and the higher one is singing at 524 Hz. The octave is the repetition tool that builds a scale.

Apart from the periodic characteristics a scale presents, there are also two more mathematical elements that have become the building blocks of music composition and understanding. Firstly, there is the tempo, that guides how fast a musical piece will be played and can vary from allegro to adagio, which means from slow to fast. Secondly, there is the rhythm, which rules everything and arranges the notes in meters. It creates the musical sentences in a way that they can be understood and enjoyed. Pythagoras' signature is found here too, as the most fundamental rhythms are the ones including 2 or 3 in their ratio and all the other rhythms are derived from combining the fundamental ones.

The rhythm is counted in pulse repetitions and the easiest way to understand this is to close your eyes and put your hand in the middle of your chest. Can you feel your heartbeat? When you are calm, your heart beats at 60 – 80 pulses per minute and its rhythm pattern is like the one of a drum. Music is inside you and so is math!

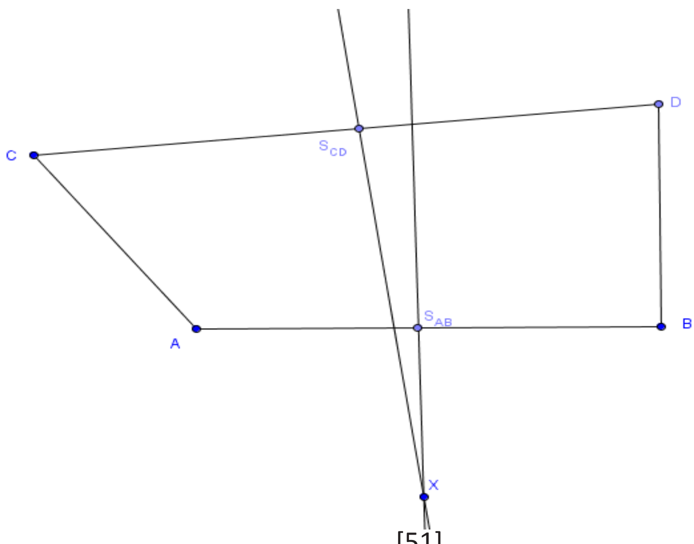
21. Where is another possibility?

When learning about the congruent theorems at school, the teacher showed us a problem that really confused us at the end of the lesson. It took me a rather long time to find the correct answer, and I will now show you the problem as well as the way I managed to solve it.

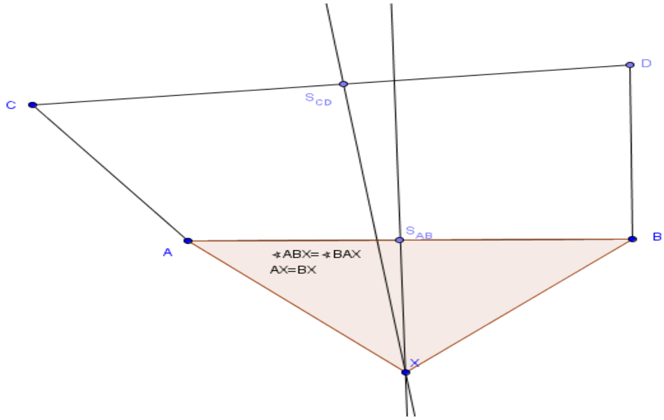
The teacher's assignment appeared to be simple. During the presentation, the presenter will use a set of pictures.

Let us draw a line segment AB and two rays that form different angles with the line segment. The best scenario would be if one was a right angle and the other obviously different /shown in the picture/.

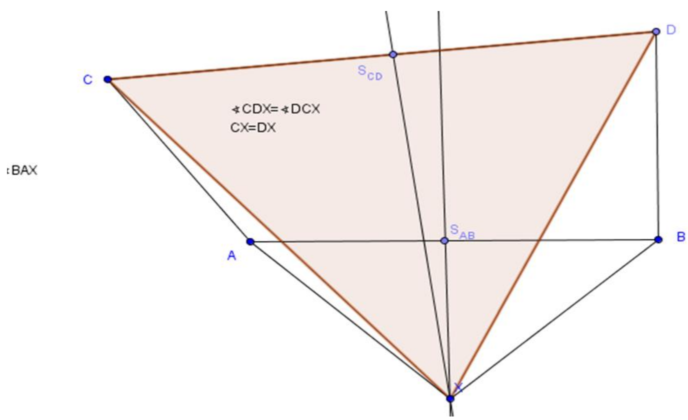
Let us mark C and D as the points on the rays whose distance from the end points of AB is the same /see the picture/.



Let us draw perpendicular bisectors of line segments AB and CD. We will mark X as their point of intersection /see the picture/.

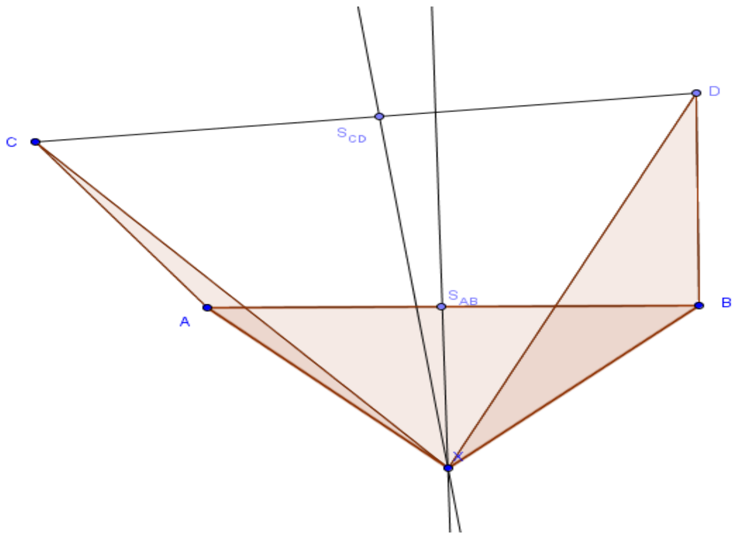


As X belongs to the bisector of AB, the triangle AXB is isosceles /see the picture/. This means that AX=BX, but also that the angle ABX is equal to the angle BAX.

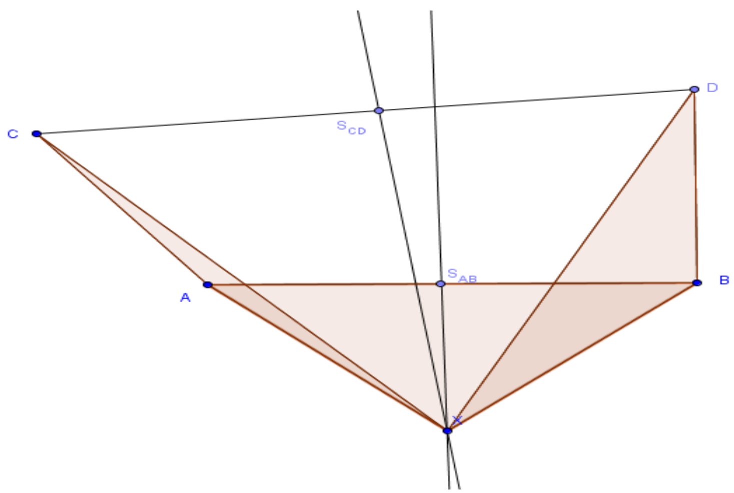


However, the point X also belongs to the perpendicular bisector of the line segment CD and therefore the triangle CXD is isosceles/see the picture/. Thus, it also holds that CX=DX.

Now we know that all three sides of the triangles CAX and BDX have the same length and therefore these triangles are congruent.



But it means that not only do the angles ABX and BAX have the same size, but also that the angles CAX and BDX have the same size. Furthermore, it follows from it that the angles CAB and DBA have the same size.



The teacher finished here and asked us to find out what is wrong with it.

While it is obvious that these angles cannot have the same size, the task was to find out where the mistake was in the previous considerations.

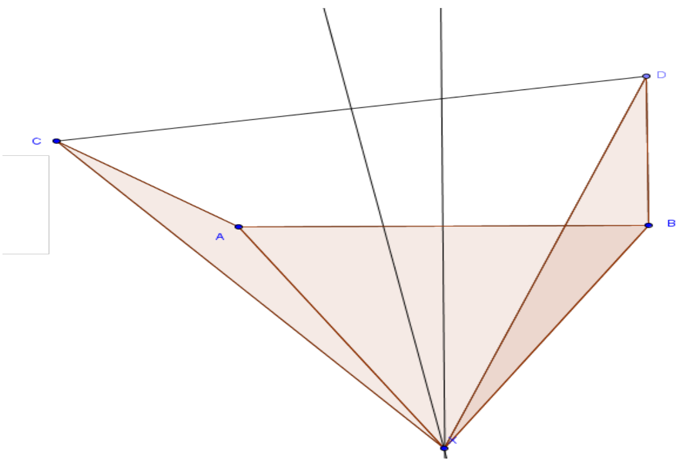
I went through it at home and could not find any mistake. Everything looked so correct. Then I had the idea that the point of intersection could be somewhere else. Thus, I drew the whole picture so that the point X would be above the line segment AB. But then, /shows the picture and repeats the previous procedure until the previous conclusion and demonstrates it on the picture/.

Thus, moving the point X above the line segment does not work.

Later I had another idea. What if the bisectors do not intersect by any means?

I drew it all, but then they would have to be parallel, therefore AB and CD would also have to be parallel and both angles would have to have the same size. Even here, I was not successful.

I went through it once more /repeats all statements again /. I did not find any mistake. I decided to draw everything accurately and found out where the mistake was /see the picture/.



Finally, I was right. The problem is in the point X but it was not above. On the contrary, it was below.

This task taught me that rough drafts can be very confusing and that it is necessary to check any consideration. It is also interesting that some mathematicians have such a sense of humor that they come up with similar riddles, like the one I have just shown you.

22. Irrationality of square root of 2

"All things are numbers" was Pythagoreans's philosophy, but they believed that only positive rational numbers could exist in the world.

The problem of the length of the diagonal of a square, led one of Pythagoras' disciple , Hypassis of Metaponte to his death, because he discovered that this length was not a rational number. Pythagoreans were so terrified by the idea of incommensurability and the existence of such numbers, that they threw Hipassis overboard on a sea trip, and kept the existence of irrational numbers in the secret of their sect.

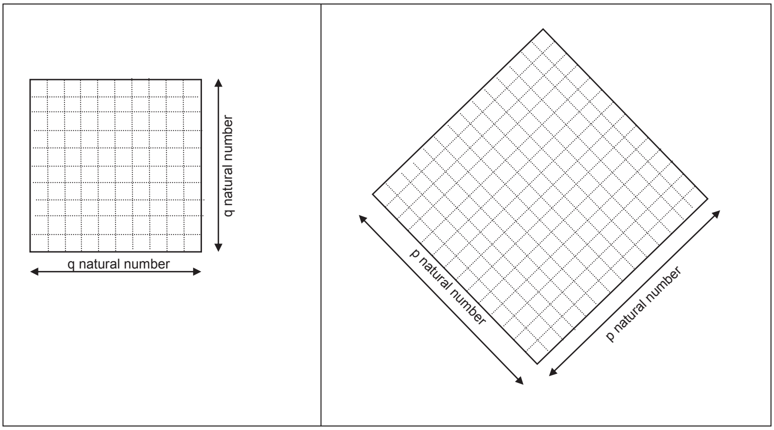
Let's have a look to this mysterious and dangerous number:
Can't we really write this number as a rational number?

<u>Hypothesis:</u>
Let's suppose that square root of 2 is a rational number, i.e it could be written p'/q', where p' and q' are natural numbers. This fraction can be reduce to an irreducible fraction p/q.

Let's assume thus, that square root of 2 can be written as the ratio of two relatively prime numbers, p and q.

The student shows two squares, and by superimposing them, can show that the area of the second one is twice bigger than the first one:

Let's have now a look at these two following squares to visualize the situation: the largest square has been built like this: its side p is the diagonal of the first one, which side is q.



Our hypothes is : $\frac{p}{q} = \sqrt{2}$ with p and q relatively prime numbers, but we can also observe this result with our two squares:

Indeed, thanks to Pythagoras' theorem we have: $p^2=q^2+q^2$, $p^2=2q^2$,
$\frac{p^2}{q^2} = 2\frac{p^2}{q^2} = 2$, $(\frac{p}{q})^2(\frac{p}{q})^2=2$ and then $\frac{p}{q} = \sqrt{2}\frac{p}{q} = \sqrt{2}$

By the way, we've just written the square root of 2 as a rational number.
But just have a deeply look at what we've really found:
p and q are natural numbers, that means that either they are **even** numbers, or they are **odd** numbers.

<u>If p is an even number</u>, then we can write p=2 x n

and $p^2= (2n)^2 = 4 n^2 = 2 \times (2 n^2)$ **<u>then p^2 is an even number again</u>!**

<u>Let's see what happens in the other case, that is if p is an odd number :</u>
Then we can write p=(2n) +1
and $p^2= (2n+1)^2 = 4n^2 + 4n +1 = 2 \times 2 (n^2 + n) +1$, thanks to remarkable identities
$p^2 =2 \times 2 m +1$ **then p^2 is an odd number !**

The only case where p^2 is an even number is when p is an even number as well!

-->Therefore, if the square of an integer is an even number, that means that this number is an even number.

The student shows a paper with this property written

<u>But let's come back to √2</u>

We 've showed, that p and q , natural numbers were like $p^2 = 2q^2 p^2 = 2q^2$

The student showing the paper with the property demonstrated before:

Therefore $\mathbf{p^2}$ is an even number, therefore **p** is an even number as well, *cf* the property we've just demonstrated before.

We can then write p= 2 m.
And the equality $\mathbf{p^2 = 2q^2}$ becomes $4m^2 = 2q^2$
Or, reducing, $2m^2= q^2$ -->therefore q^2 is an even number --> **q is an even number.**

We've just demonstrated that p and q are both even numbers

Which is contradictory with the fact that p and q are relatively prime numbers!
Our first hypothesis must be false!

We cannot find any positive rational numbers whose squared is two, therefore,
√2 is an irrational number

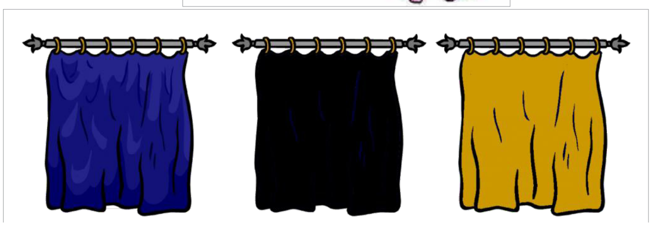
This type of demonstration is called « demonstration ad absurdum »

23. The Monty Hall Show

I'm going to present you a famous game show called "Let's make a deal" or the "Monty Hall Show":

The student can show a poster where the audience can see three doors, that can be opened, or three curtains that hide one car and two goats.

or



The presenter is Monty, he shows you three doors. Behind one door there is a car, and behind the other two doors you can find a goat. You can win what you find behind the door you choose. Let's see the chances you have to win the car!

Monty Hall asks you to pick a door, e.g, you pick the 1st door.

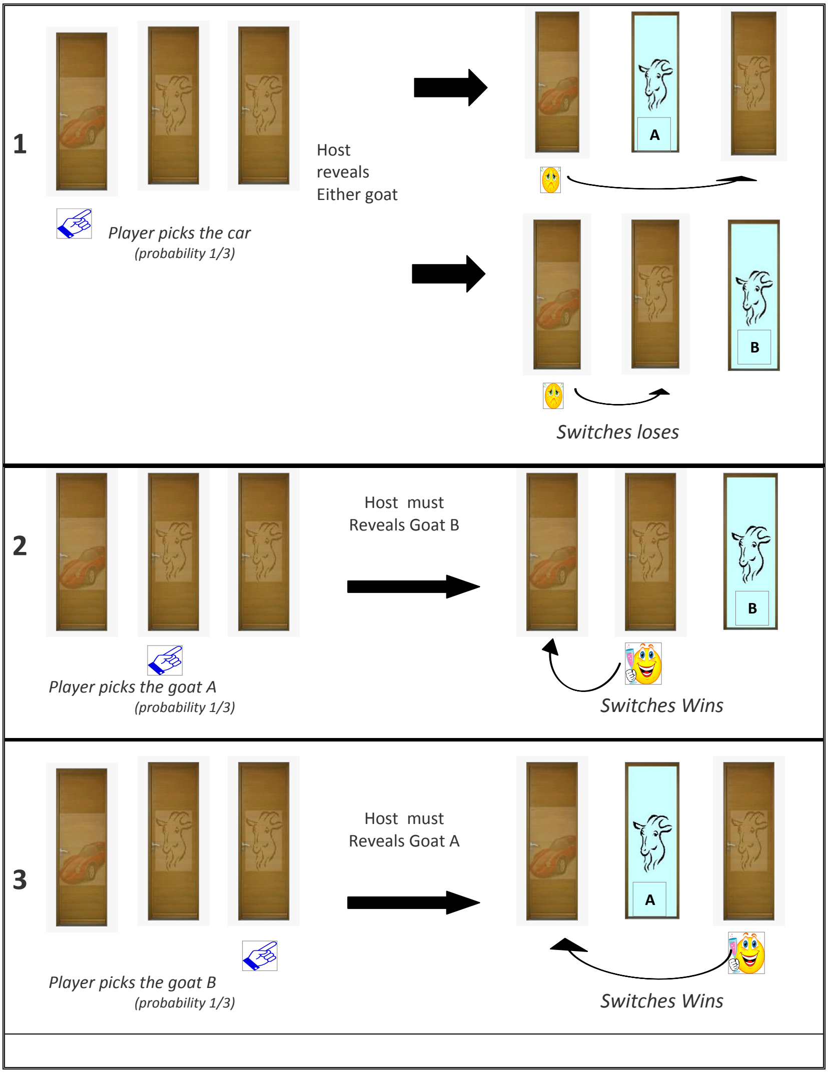
Monty Hall now opens up another door, e.g the 3rd door, behind which there is a goat. Of course, Monty Hall knows exactly where the car is, but will never open that door!

He then asks you if you'd like to stick with your first guess, the 1st door, or switch to door number 2.

The question is: is it to your advantage to switch your guess to the 2nd door? Intuitively, it seems to make no difference whether you switch your choice of door or not.
But what happens when you think about this in a probabilistic way?

The study of probabilities can help to understand the problem and find what is the better way to win the car, switching or not switching? Let's see:

The student shows a diagram :



The initial probability of picking the correct door is 1 in 3.
Actually, you have 1 chance in 3 of picking the car before any other information is given.

However, once you see a goat behind door number 3, there are just a couple of outcomes left:

1. Either you correctly picked the car the first time, which was a 1 in 3 chance, in which case switching guarantees that you will lose.
2. or you initially picked a goat, which was a 2 in 3 chance, and then by switching you will select the door with the car, as you are unable to select the goat that has already been shown to you.

Let's recap the diagram:
Option 1: switching makes you lose the car, with a probability=1/3
Option 2: switching makes you win the car, with a probability=1/3
Option 3: the same situation as option 2, switching makes you win the car, with a probability=1/3
so, if you systematically switch, you have a probability of 1/3+1/3=2/3 to win the car, whereas you have a probability of 1/3 not to win the car
Finally, we can see that the answer is that switching doubles your chances of winning the car!
Well, now, the question is, what do you really want to win: a car or a goat?

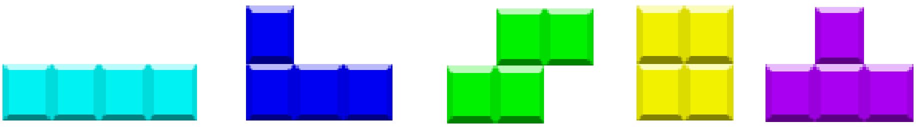
24. Playing Tetris

Last week while cleaning our house, we came across an old wooden box. When I opened it I found some old stuff that my father owned when he was in my age. There, I found some weird clothes, a watch, an old camera and an old game that my father used to play when he was young.

When I asked him what this game was he answered:

“Oh, that is one of the classic arcade games called Tetris. The version of Tetris that I used to play was played like this:

Tetriminos are game piece shapes composed of four - square blocks each. Like this:



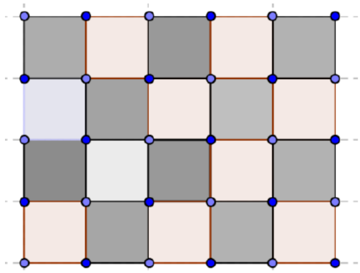
A random sequence of Tetriminos falls down the playing field.

(The student shows one of the pieces and manipulates it the same way he describes below)

The objective of the game is to manipulate these Tetriminos, by moving each one sideways and rotating it by 90 - degree units, with the aim of creating a horizontal line of 5 blocks without gaps. Now, if you think you are clever, go ahead and play this game”.

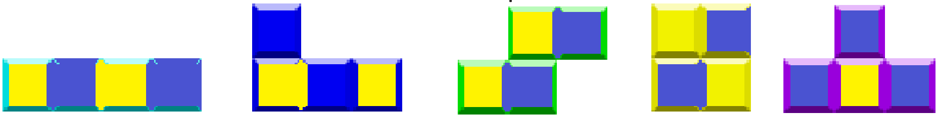
I started playing this old game and I came across a problem:

If the first five tetriminos are these (shows again the piece of paper), is it possible to cover in full 4 rows of 5 blocks like this (student shows the picture below)?



Well, let us see. In order to cover 4 lines of 5 blocks I need 20 blocks. The five shapes are made up of 4 blocks each. Thus, I have the 20 blocks that are needed and it seems possible to cover it.

However, if I colour the field in black and white, just like a chessboard, we see that there are 10 white and 10 black free spaces.



The 4 pieces have two yellow and two blue blocks each. The last piece, though, is made up of three blocks of the same colour and one block of a different colour. This means that in total, with these five pieces, we have 11 blue blocks and 9 yellow blocks, while we need 10 blocks of each colour. As a result, it is impossible to cover the field with these 5 pieces.

One thing is for sure. You don't need to make such analysis when you play Tetris but you will need it if you have to cover a room with tiles.

25. To tell a lie or to tell the truth? That is the question!

Once upon a time, a mathematician was travelling to the Arabic countries to find the origin of a method to be used in solving equations. While he was there, he was taken for a spy and was sentenced to death.

He was a mathematician, and was trying in vain to explain that all he was looking for was the secret of the formula to solve quadratic equations, i.e. the formula which was known by Babylonian and Egyptian mathematicans, but was forgotten throughout the centuries. Perhaps the mathematical scripts kept in the Arabic language would reveal the secret and it would be an advantage if it were known by all people instead of it being kept hidden.

Unfortunately, he could not change the death sentence, but the Arabs decided to tell him to choose the way in which he would prefer to be executed. He was asked to tell a lie or the truth. In the first case, if he told a lie, he would be hanged like thieves. In the second case, if he told the truth, he would be decapitated like the brave enemies.

He then said a sentence, and the Arabs released him. He was given access to the formulas previously kept as secret. He became respected for his wisdom.

What was the secret of the sentence he said?

Did he say a mathematical theorem? How did he escape? We know that Arabs are good mathematicians and always keep their promises.

The man said: "I will be hanged".

If this sentence was considered to be true, he should have been decapitated, but then the sentence would not be true anymore, and he would have to be hanged.

If the sentence was considered to be a lie, he should have been hanged. But again, in this case, the sentence would become true, and the Arabs would have to keep their word and decapitate him, as they had promised.

As a consequence, the Arabs did not kill him. Instead, he became respected for his wisdom, and later on, the formula to solve second order equations (SZÓREND) became a famous start for Al-Jabr, which means Algebra nowadays.

At the same time, this story is a good starting point for talking about modern logic where we use the expression of proposition more. That is a declarative sentence which is either true or false. It can be considered a logical proposition only and only if its logical value is not changing. Of course, there are other sentences, like logical functions: sentences which may contain logical variables, say x, and depending on the value of x, they become true or false. Example $x^2 =4$ is true for x=2, and x=-2, but false for any other value of x.

26. Pigeonhole principle

This principle got its name from a simple case which you can easily understand.

Let us imagine that we have three pigeons and only two pigeonholes and we want to put the pigeons in the pigeonholes. There is no doubt that we need to put at least two pigeons in one of the pigeonholes, but we do not know in which one. There are several choices: to put all three pigeons in the first or in the second pigeonhole, or to place two of them in the first and one in the second, or vice-versa, i.e. one in the first and two in the second. In any of the cases, it is excluded to have only one in each pigeonhole, and to place all pigeons, as the number of pigeons is greater than the number of pigeonholes. Consequently, you need to use the same pigeonhole (one of them, but we do not know which one) for at least two pigeons.

Let us take a similar problem but in another context. Imagine you have white and black socks in a box, and you pick them up one-by-one. You have a pair of socks: black or white. What is the minimum number of socks you need to take out from the box?

The right answer is three. This is obviously a different problem but with the same principle. Now imagine that the two colours are the two pigeonholes and the socks are the pigeons this time. If the number of socks is greater than the number of colours, you need to repeat the same colour. We do not know which colour, but we know it MUST HAPPEN.

We can now formulate the principle in a more general form. If we have n pigeonholes, but at least n+1 pigeons, you will have to put two pigeons in at least one of the pigeonholes. If we did understand the principle, we would be able to discover it in various situations, for various problems.

Do you call your classmates on the phone? Of course. Every day you and your colleagues make dozen and dozen, or even more phonecalls. But did you know that each day, without exception, there are at least two colleagues who talked on the phone with exactly the same number of classmates that day? Could you prove that?

Let us think a bit: what are the possible numbers of chats? Of course, there could be someone who did not call any of the classmates, but this is not certain. At the same time, any of the classmates could have n-1 calls, at most, as chatting with yourself would be nonsense. But the two extreme situations would exclude each other. If someone had no calls at all, then nobody could chat with ALL n-1 classmates, and vice-versa. If somebody had the maximum number of phonecalls, then he/she did speak to EACH of his/her classmates and no person could have 0 chats. In other words, if we consider the numbers of all possible chats, this set of numbers cannot contain 0 and n-1 at the same time. This means that we have only n-1 choices for the possible number of calls (0, 1, 2, and so on to n-1), that is n but at least one of the two extremes is missing. Hence, we can have only n-1 different figures to be “shared” n people, so again the pigeonhole principle says that at least one figure must be repeated. As a consequence, every day we need to have at least two of the classmates who had the same number of chats that day.

Think over some other situations in which the pigeonhole principle works!

Could you imagine the size of a school where at least TWO pupils MUST exist who have the same birthdate?

Is it true that at Oxford University there MUST be at least two students with the same initials (e.g. John Brown and Jessica Bishop)?

Try to formulate similar sentences based on the pigeonhole principle!

27. The Tower of Hanoi

The Student gets up on stage and shows a small wooden game, called the Tower of Hanoi.

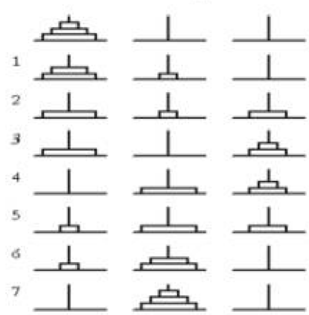
"Today I will speak to you about the Tower of Hanoi. It contains three vertical bars and a given number of discs of increasing diameter, placed one over the other on the first bar, with the largest at the base and the smallest at the top. The problem is to move them one-by-one, using the two other bars as well, but in every step you can move only one disc, and you can only put a smaller over a larger one. With one disc you obviously only need 1 step, with two discs you would only need 3."

The Students shows the moves for one, then for two discs.

"The legend says that in a faraway monastery in Hanoi, there is such a tower which contains 64 discs and whenever somebody manages to solve it, that will be the end of the world…"

"The reality is that if you know how to solve the problem, even without making any mistake, just the minimum number of necessary steps, it will take you a huge number of moves, exactly **2^{64}-1 steps**. To imagine that, you should be able to make a correct move every second. This means you would need about 585 000 000 000 years which is 585 billion years. Of course, if you only have a small number of discs, the problem is rather simple, as you have seen for one or two discs."

"Let us see how it works for three!"



"The strategy to solve it is rather simple. You need **3** steps to move the upper two discs in a "parking position", the third bar, then move the last disc to the bar in the middle, say the final position and then move the parking discs over it. Again, you need **3** moves, that is altogether **3+1+3= 7**, but 7 is $\mathbf{2^3-1}$, so our clue is right.

In fact for one or two discs we can remark that similarly the "formula", our clue is true, because $\mathbf{1=2^1-1}$, and $\mathbf{3= 2^2-1}$

Could we prove that this is what generally happens?

I guess we can. It happens all the same, if we now have our "clue" for 3 discs, we can start with four. Again, it is easy to observe that we need to move all three on the top to a "parking" position as before, then move the last one, the largest to the final position, and again move the parking ones. The number of steps we need is $\mathbf{2(2^3-1) +1=2^4-2+1=2^4-1}$.

The number of steps we need is again **twice the number of steps for the previous case, plus one.**

When using a mathematical formula, if you have the number of $\mathbf{2^n-1}$ of steps for a given number **n** of discs, then by adding one more disc, n+ 1 disc you will need twice the number of the previous steps, plus **1.** $\mathbf{2(2^n-1)+1=2^{n+1}-2+1=2^{n+1}-1}$. That means the formula is TRUE for **n+1** as well. You got it!!

The method used here is called mathematical induction, and it is a basic principle we use in properties depending on natural numbers."

The student will find a solution for four or even five disks. He will then loudly count the steps.

28. Clever squaring

In mathematical computations it is important to know some tricks and easy methods in order solve the most common computations. Of course, the use of pocket calculators might be a solution, but sometimes you may not have access to such a tool, or you simply want to avoid using one for simple calculus. There are plenty of examples such as calculus related to the Pythagoras' theorem, solving second order equations and computing the area of squares.

The problem is when computing the square of two digit numbers.
We usually multiply them as shown below:

57 x 57=	we make the whole multiplication in the usual way,
57 x 57 =	firstly we multiply by 7, the first number
399	then multiply by 5, and place the result one digit to the left
285	then add the two rows, **completing the last, missing digit by 0**
3249	

However, there is another way this can be done.
Take the square of the first digit, and complete it by two 0-s
25**00**
Then take the double of the product of the two digits, and complete it by one 0 on each side, 2 x 5 x7 = 2 x 35 = 70
0700
Now take the square of the second digit, and complete it in front with two 0-s, that is
0049,
Then finish by adding the three results:

25**00**
070**0**
0049
3249

Of course, if you practice this method, you will observe, that the 0- are only places, so the final result can be written in the following form as well:

57^2 =
25
 70
  49
3249

Similarly, you can multiply any two numbers with two digits!

What is the mathematical background of the method? It is easy to understand it because each number with two digits can be written in the form 10a+b, where a and be are the digits of the number. In the previous example, a=5, b=7, because 57 can be written as 5 x 10 + 7 = 50 + 7 = 57.
But from algebra we know that the square of such an expression, a binom 10a+b , can be easily computed algebraically.
$(10a+b)^2 = 100a^2 + 20ab + b^2$, thus the "clever squaring formula" will be
$100a^2+$
.20ab+
....b^2

In a formal way, if xy are the two digits of a^2 (if a=1,2, or 3, x=0), similarly if zuv are the digits of the term 2ab, and st the digits of b^2, then the formal calculus will be:
xy00+
0uv0
00st
or the same without the zeros:
xy+
.zuv
....st

It is easy to see that the first row ends in two zeros while the second row in one zero. So if we order them as shown, we may even skip the zeros. We need to concentrate on the ending of the numbers, because the forms xy, uv or st can be numbers of one or two digits (even 3 as for the middle one zuv), as seen in the next example

$83^2 =$ 64 48 9 **6889**	$27^2 =$ 4 28 49 **729**	$31^2 =$ 9 6 1 **961**	$89^2 =$ 64 144 81 **7921**

The student will solve the squaring for 2-3 other examples.

29. The circle and the others

(The student comes to stage holding a hula-hoop pretending to be a circle)

I just got away. A mad quadrilateral was chasing me. He was trying to square me.

They have been trying that for Centuries now, but unsuccessfully. I thought that they gave up but it seems that they did not.

He was trying to get close to me, very politely at the beginning. He was saying that my absolute symmetry inspired him and that my bright arrays were bringing light to his loneliness.

He was trying to touch my essential chords. He used to say, I want to be friends, I want to get inside your circle and he was getting closer to me.

Then he started to become tangential to me (the student presents that her body is the circles tangent).

Then he told me without any shame, “What a nice circumference, I would love to square it.”

I don’t know what to do. I have become a target. The other day one crazy straight line was after me. I got tired trying to avoid her.

She came up to me from nowhere. As I was making my round on a sine curve suddenly, I ended up making rounds on that straight line. That crazy line became my tangent and she was only a radius away from my centre. But what is worse is that she thought that the one common point we shared was not enough. She tried to become a carrier of one of my chords. And jumping from one chord to the other she became a carrier of my diameter. That means that she passed from my centre and she became a line of symmetry. Can you imagine she had cut me in to two equal parts?

She would not get away from me. I was trying so hard to get rid of her. In the end, she tried to get the upper hand, “You are very unsocial, you are a vicious circle.”

It is a fact that all rectilinear shapes want to interact with me. For a polygon to get the title of regular, it firstly has to have its edges on my circumference and also, all my arcs that form from two consecutive edges have to be equal.

And do not forget that every time the regular polygons have a conference, they align one after the other with the triangle going first, then the square, then the pentagon and the number of edges increase when trying to use the limit to become a circle. But they don’t understand that you are born a circle and you do not become one.

Well I think I know the answer. I have so many beautiful properties and that is

why everybody wants me.

30. The loneliness of the top

(The student pretends to be the nonprime number 39)

When I see this number (student shows number 2) it makes me dis-number.

As you see, two is the structural unit for the even numbers and not only that; it is the only even number that is prime.

What I do not understand, is why does someone have to be proud because he is a prime, or why he is proud of the fact that his only dividers are himself and one.

Thank god Euclid for that. From the moment that Euclid had proved that every number could be written in a unique way as a multiplication of prime numbers, the primes have overestimated themselves.

Euclid has also proved that there is an infinite number of primes. Mathematicians are scandalously in favor of them.

And as if this was not enough, Eratosthenes came along with his sieve. He threw all of us in the trash just so that he could separate the prime numbers from all the other natural numbers.

And after the renaissance in Europe, a new mania about the prime numbers came along. Mathematicians were looking for a formula that generates prime numbers. The best mathematical minds were working on that, Fermat, Gauss, Euler, Riemann… The denial from the part of the primes to reveal their causality makes them the most important mathematical matter in Europe.

To be honest with you I am a little jealous of these Primes. Even though I consider them as servants. Why? Well it's simple. They undergo multiplication to make... me! 39 = 3*13

Not only that, I couldn't bear their loneliness. Do you think that if Eratosthenes was using his sieve for numbers between 1 000 000 to 1 000 100 he would have many primes left in his sieve? Just to inform you, if the prime number 1000 019 was to make friends with his next door prime number he would have to travel for a long time to find 1000 079 while for much bigger numbers the distance is even greater.

Ahhh...The absolute loneliness. I feel very lucky to be a composite number.

31. The Pigeonhole Principle

(The Student gets up on stage and starts counting people in a row with more than 12 people sitting).

"I bet that in this row there are at least two people that have the same horoscope. Let's see! How many of you are Pisces? Aquarius? *(He proceeds to the following horoscope until two people have the same horoscope).*

"Now do you know why I was certain about my statement?

Well I am not an astrologist but I do like mathematics.

It's all based on the pigeonhole principle. '

In mathematics, the pigeonhole principle states that if n items are put into *m* pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item.

The first formalization of the idea is believed to have been made by Peter Gustav Lejeune Dirichlet in 1834 under the name *Schubfachprinzip* ("drawer principle" or "shelf principle"). For this reason it is also commonly called Dirichlet's box principle.

That means that if I have 8 pigeons and only 7 pigeonholes and I have to place all of them in the holes, then I would place the first 7 pigeons in the seven different holes. As a result, there is no hole left for the last pigeon so I have to place it in one of the holes that have already been used. Therefore, one of the holes will have 2 pigeons.

Ok so what? Are there other applications of this principle? Well let's see.

I am certain that in New York City, there are two non-bald people who have the same number of hairs on their head.

The human head can contain up to several hundred thousand hairs, with a maximum of about 500,000. In comparison, there are millions of people in New York City. Consequently, at least two of them must share the same number of hairs.

Here is another one. If you have 10 black socks and 10 white socks and all of them have the same design, and you are picking socks randomly, how many socks do you need to pick in order to find a matching pair? The answer is three.

If I take 1 sock then I don’t have a pair. Then I pick another sock. If it’s the same colour as the first one then the problem is solved. However, if it’s not the same colour, that means that I now have one black and one white sock. Now, if I pick a third sock, then the new one will either be black or white and I will therefore have a matching pair.

Another way of seeing this is through the pigeonhole principle. I have three socks and only two colours, therefore; at least two must be of the same colour.

So, next time you are in a place with 13 people, you will know that at least two of them have the same horoscope.

32. The story of the ladybirds

Once upon a time there was a wonderful place with many colourful flowers, a warm sun and a gentle breeze. This great place was inhabited by … ladybirds.

They were all equally cheerful, hardworking, helpful and different by … can you guess? Colour … and the number of dots on their back. Yet an input rule was required:

Some suggested that they should be grouped by colour as there were red, orange and black ladybirds. So, the red (which were also the most numerous) entusiastically agreed with the idea and refused to accept the black ladybirds in their club.

This immediatelly sparked an argument. The black accused the red of trying to keep them away from public life.

The orange struggled to reconcile both sides, explaining that this could not stand for a valid criterion, as any creature, regardless of the colour, is equally beautuful and valuable. This caused total chaos. The ladybirds became increasingly louder and even began to jostle, until an orange ladybird cried out the saving solution: they would be grouped by the number of the dots on their back!

The red ladybirds looked at the black ones, the latter exchanged the looks, then both sides started counting each other's dots and expressed their satisfaction: they would be grouped by the number of the dots.

The Club of the Even Numbers was set and admittance was guaranteed only to those who possesed an even number of dots (i.e. divided by 2).

There were many ladybirds willing to join the club who also matched the requirement. Therefore, it was decided to open two more clubs of the same type/profile. The Squared Club (the total number of dots divides by 4, 2 times by 2, therefore the last two figures of the number must divide by 4) and the Club of the Eights (divide by 8, 3 times by 2, therefore the last three figures of the resulting number divide by 8).

What about the Odds? Where should they go? They thought it over and decided to set the club of the Thirds, where admittance was guaranteed to those ladybirds whose dots could be grouped by 3. Entrance to the club was quite crowded so the ladybirds were becoming increasingly impatient.

A new filter-rule was necessary, a more efficient one. That was when a smart ladybird noticed that if you add up the figures of the total number of dots and the number obtained divides by 3, then the initial number divides by 3. And, to be able to group even better, the New Kit was founded and only the ladybirds whose number of dots divide by 9 were allowed to enter (it was also the clever ladybird who noticed that the same rule could be applied if the sum of the figure divides by 9, then the number itself divides by 9).

And those who had the right number of dots, so as to group by 6, discovered that they could join two clubs: The Evens and The Thirds. Have you figured out why?

The number of data divides by both 2 and 3. How lucky they were, weren't they?

Yet, there were ladybirds who could not join any of the dules. So they gathered together, calculated and came up with another solution: The Club of the Fifths, comprising the ladybirds, whose dots can group by 5. How were they supposed to count more easily? It was immediately obvious that all ladybirds, whose dot numbers end in 0 or 5 can enter the club. And to be closer together, those whose last figure was 0 formed The Fellowship of the Tenths.

Every night, the ladybirds would polish their wings, spray a little perfume and fly away to their group's meeting place. And there was great joy, music and laughter floating in the air like balls full of energy…

Yet there were still ladybirds left out. Most of them had 7 dots on their wings but there were others which had 11, 13 and 17 dots. These ladybirds lay there in the grass worried and upset. They had knocked on all doors but found no place to match, as none of their fellows would welcome them. Feeling rejected and misunderstood, they decided to form a group which would have special properties so as to include them all. In this club they would read a lot, talk about life, the good and the evil, and they would especially try to find out what kind of evil lay out there, beyond the things they had already known.

Being different from others is a privilege, as you can explore and better learn your profound interior world! Any ideas on what their club should be called?

The club of the Prime Numbers.

33. Where there is an X...there pops in 0, too!

I believe we all have played the famous game of Tic Tac Toe at least once in our lifetime. At first sight, this looks like a simple child-oriented game. However, it involves mature strategic approach and foreseeing the opponent's possible turns.

Tic Tac Toe is a two-person game and its classic version is played in a 3X3 square. The rules are simple: the two players take turns to fill in a box of the square with the chosen mark: X or 0.The winner is the one who succeeds in marking three boxes in a straight line: horizontal, vertical or diagonal. If all the boxes are marked but neither of the two players has a straight line then the game ends in a draw.

Let us make a brief analysis of the game. Even in this simple 3X3 version of the game, the number of possible turns is very big: 15 210 different options for the first five turns only! Yet, a smart player may become invincible if he/she follows a few basic schemes. If both players are aware and apply these strategies, then we have a draw.

For example, the player who starts first has three options: the box in the centre, in one of the corners or in one of the sides. From all these, the most efficient option is the corner, as the opponent could avoid being trapped at the next turn through only one of the eight possible options: the centre. Could you tell what the other strategies of the game are?

A more interesting version of the game is the one with six movable counters. This game was widespread in ancient times among the Chinese, the Greeks and the Romans. It is also played on a 3X3 square with three counters of one type and three counters of another type (two-sized coins can also be used). The players take turns to put one counter at a time on the playing board until they finish them. If neither of the players has won by placing their three counters in a line, the game continues by moving (in turns) one counter, only in a neighbouring horizontal or vertical box. The game goes on until one of the players wins. Unlike the classic Tic Tac Toe version, in this case the first player surely wins if he/she places the first counter in the centre (and applies the proper strategies). That is why this turn is usually forbidden.

If you are fond of space geometry, there is also a tri-dimensional version of the classic Tic Tac Toe. It is played in a 3X3 cube and the winner must fill in a line of three on whichever cube side, even on one of the cube diagonals. This version is indeed a space adventure! And if you are keen on an even greater challenge, you may play the version suggested by a humorous American: the first player who gets a line of three *loses*!

34. How to generalise? What to generalise? The case of Pythagoras' theorem

Take a theorem, take the most popular theorem in mathematics: the Pythagoras' theorem.
It has many equivalent forms, maybe hundreds. However, the most well-known formulation is perhaps the one which states that:

In a right angled triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides, the legs.

We may write it as a mathematical relation: $\mathbf{d^2 = a^2 + b^2}$, where we denote by **d** the lenght of the hypotenuse, and by **a**, and **b** the lenght of the legs respectively.

We know many other equivalent formulations, like the area of the square (half circle, etc) built on the hypotenuse being equal to the sum of the area of squares (half circles, etc.) and built on the other two sides, the legs.

Let us take the following reformulation of the famous theorem. Take a rectangle which has two sides equal to **a** and **b**.

Then the diagonal **d** of the rectangle is expressed by the same relation: $\mathbf{d^2 = a^2 + b^2}$**.**

Now, what is the most natural generalisation of a rectangle? Maybe the solid which we call a right rectangular prism, also called a cuboid, or informally a rectangular box (or brick).

Imagine such a solid, which has the three sides equal to **a, b,** and **c** respectively.

Now can you find an expression for its diagonal **d**? Of course you can!

It is given by the relation $\mathbf{d^2 = a^2 + b^2 + c^2}$**.**

Can you go further? Can you imagine a four or more dimensional "brick"?

Of course, you can hardly see" its form, as we are "restricted" in our three dimensional space, but we CAN say the lenght of the diagonal **d** of such a **4** (or even **n**) dimensional "brick" which has got its "dimensions" (sides) equal to a_1 , a_2 , a_3 , a_4 , (and so on… a_n) from the relation:

$$d^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 \ldots\ldots\ldots (\ldots\ldots + (a_n)^2)$$

35. How to find a rectangle when building your house? The application of Pythagoras' theorem

John's family is about to build their own house in a small village. They need to design the foundation of the house. John's father asked his son to invent something to be able to draw a right angle, to ensure that the foundations of the walls keep the perpendicularity of the walls. John was good in Mathematics, so what he did apply is the most popular theorem in mathematics: the Pythagoras' theorem. He took 3 wooden slats, cut three pieces and measured their length very carefully in order to be of 3m, 4m, and 5 m. He made a big triangle with these three slats and he was sure he had solved the problem; he got a right angle triangle. He gave it to his father, who could now make up the foundation being sure that the walls would be projected perpendicular to each other.

Remark: For the MathFactor presentation, it is sufficient to make up a model with the lengths 30cm, 40cm and 50 cm, to show it during the presentation.

It has many equivalent forms, maybe hundreds. However, the most well known formulation is perhaps the one that states that:

In a right-angled triangle the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides, the legs.

We may write it as a mathematical relation: $\mathbf{c^2 = a^2 + b^2}$, where we denote by **c** the length of the hypotenuse, and by **a**, and **b** the length of the legs respectively.

The reverse of the theorem is true as well. That is, if in a triangle ABC the three sides satisfy the above relation, the triangle is a right angled triangle, with the hypotenuse **c**, and legs **a** and **b**.

If **a, b** and **c** are natural numbers and they do satisfy the relation $\mathbf{c^2 = a^2 + b^2}$, we would then call **a, b, and c** the **Pythagorean triple.** The most known Pythagorean triple is **3, 4, 5**, which is applied for the given practical case. It is easy to show that any number greater than 3 can be one of the numbers of a Pythagorean triple.

We know many other equivalent formulations of the theorem, like that the area of the square (half circle, etc.) built up on the hypotenuse is equal to the sum of the areas of squares (half circles, etc.) built on the other two sides, the legs.

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